

Vectors in 3-D: dot and cross products, lines and planes

Ten problems on vectors in three dimensions: lengths, the dot product and the angle between two vectors, work and the component along a direction, the cross product and the area of a parallelogram, a triangle's area and the plane through three points, the volume of a parallelepiped from a determinant, a line meeting a plane, the distance from a point to a plane, the angle and line of intersection of two planes, the distance between skew lines and a linear classifier's signed distances, with worked solutions and the mistakes of the middle sign of the cross product, dividing by squared lengths, forgetting to divide by $|n|$, a plane through the wrong point and the product of lengths as a volume.

BEFORE YOU START

Two products do most of the geometry in three dimensions. The dot product $u \cdot v$ measures how much two vectors point the same way: it gives lengths, angles, projections and the work done by a force, and a zero dot product means perpendicular. The cross product $u \times v$ builds a third vector perpendicular to both, whose length is the area of the parallelogram they span. A plane is described by a point and a normal vector, a line by a point and a direction, and every distance between them is a dot product with a unit vector. The same formulas reach machine learning unchanged: cosine similarity is the angle formula, and a linear classifier's decision boundary $w \cdot x + b = 0$ is a plane with normal w . These ten problems compute angles, areas and volumes, write lines and planes, find where they meet and how far apart they are, and end with the signed distance of a point to a decision boundary.

- For $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$: $u \cdot v = u_1v_1 + u_2v_2 + u_3v_3$ and $\|u\| = \sqrt{u \cdot u}$.
- **Angle.** $u \cdot v = \|u\| \|v\| \cos \theta$, so $\cos \theta = \frac{u \cdot v}{\|u\| \|v\|}$, with $0 \leq \theta \leq \pi$. $u \cdot v = 0$ exactly when $u \perp v$ (or one of them is 0).
- **Component and projection.** The component of v along u is $\frac{u \cdot v}{\|u\|}$ and the projection is $\frac{u \cdot v}{u \cdot u} u$.
- **Cross product.** $u \times v = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$. It is perpendicular to u and v , follows the right-hand rule, $v \times u = -(u \times v)$, and $\|u \times v\| = \|u\| \|v\| \sin \theta$ is the area of the parallelogram on u and v .
- **Triple product.** $a \cdot (b \times c) = \det[a \ b \ c]$, the determinant with a, b, c as rows, and $|a \cdot (b \times c)|$ is the volume of the parallelepiped on a, b, c .

- **Lines and planes.** The line through P with direction d is $r(t) = P + td$. The plane through P_0 with normal $n = (a, b, c)$ is $n \cdot (x - P_0) = 0$, that is $ax + by + cz = n \cdot P_0$. The distance from a point Q to the plane $n \cdot x = k$ is $\frac{|n \cdot Q - k|}{\|n\|}$.

Problems

1. For $u = (1, 2, 2)$ and $v = (3, 0, 4)$, find $\|u\|$, $\|v\|$, $u \cdot v$ and the angle between them. difficulty ·

2. A force $F = (2, -1, 3)$ moves an object along the displacement $d = (4, 2, 1)$. Find the work done, the component of F along d and the projection of F onto d . difficulty ·

3. For $u = (1, 2, 2)$ and $v = (3, 0, 4)$, find $u \times v$, check that it is perpendicular to both, and find the area of the parallelogram and of the triangle on u and v . difficulty ·

4. Find the area of the triangle with vertices $A(1, 0, 0)$, $B(0, 2, 0)$, $C(0, 0, 3)$, and the equation of the plane through them. difficulty ··

5. Find the volume of the parallelepiped on $a = (1, 0, 1)$, $b = (2, 1, 0)$, $c = (0, 1, 3)$, and the volume of the tetrahedron with the same three edges from one vertex. difficulty ··

6. Find the line through $P(1, 2, 3)$ and $Q(4, 0, 5)$, and the point where it meets the plane $x + y + z = 12$. difficulty ··

7. Find the distance from $Q(2, 1, 3)$ to the plane $2x - y + 2z = 4$, and the point of the plane nearest to Q . difficulty ··

8. Find the angle between the planes $x + y + z = 1$ and $x - y + 2z = 3$, and their line of intersection. difficulty ··

9. Show that the lines $L_1 : (1, 0, 0) + s(1, 1, 0)$ and $L_2 : (0, 0, 2) + t(0, 1, 1)$ do not meet and are not parallel, and find the distance between them. difficulty ···

10. A linear classifier predicts the positive class when $w \cdot x + b > 0$, difficulty ...
with $w = (2, -1, 2)$ and $b = -3$. Find the signed distances of $x_1 = (2, 0, 1)$ and $x_2 = (0, 1, 0)$
to the decision boundary, and the point of the boundary nearest to x_1 .

Answers

1. $\|u\| = 3, \|v\| = 5, u \cdot v = 11, \theta \approx 42.83^\circ$
2. $W = 9$, component $\frac{9}{\sqrt{21}} \approx 1.964$, projection $(\frac{12}{7}, \frac{6}{7}, \frac{3}{7})$
3. $u \times v = (8, 2, -6)$; parallelogram area $2\sqrt{26} \approx 10.20$, triangle area $\sqrt{26} \approx 5.10$
4. Area $\frac{7}{2}$; plane $6x + 3y + 2z = 6$
5. Parallelepiped 5, tetrahedron $\frac{5}{6}$
6. $r(t) = (1, 2, 3) + t(3, -2, 2)$; it meets the plane at $(7, -2, 7)$
7. Distance $\frac{5}{3}$; nearest point $(\frac{8}{9}, \frac{14}{9}, \frac{17}{9})$
8. $\theta \approx 61.87^\circ$; line $(2, -1, 0) + t(3, -1, -2)$
9. Distance $\frac{1}{\sqrt{3}} \approx 0.5774$
10. Signed distances 1 and $-\frac{4}{3}$; nearest boundary point $(\frac{4}{3}, \frac{1}{3}, \frac{1}{3})$

Worked solutions

Problem 1

For $u = (1, 2, 2)$ and $v = (3, 0, 4)$, find $\|u\|$, $\|v\|$, $u \cdot v$ and the angle between them.

$\ u\ = \sqrt{1 + 4 + 4} = 3$ and $\ v\ = \sqrt{9 + 0 + 16} = 5$	Square, add, take the root.
$u \cdot v = 3 + 0 + 8 = 11$	Multiply matching entries and add.
$\cos \theta = \frac{11}{3 \cdot 5} = \frac{11}{15} \approx 0.7333$	Divide by the product of the lengths, not their squares.
$\theta = \arccos(0.7333) \approx 0.7476 \text{ rad}$	A positive dot product means an acute angle.
$\ u\ = 3, \ v\ = 5, u \cdot v = 11, \theta \approx 42.83^\circ$	In machine learning $\frac{11}{15}$ would be the cosine similarity of u and v . Dividing by $\ u\ ^2\ v\ ^2$ instead gives 87.2° (Mistake 2).

Problem 2

A force $F = (2, -1, 3)$ moves an object along the displacement $d = (4, 2, 1)$. Find the work done, the component of F along d and the projection of F onto d .

$W = F \cdot d = 8 - 2 + 3 = 9$	Work is force times displacement, counting only the part of the force along the motion.
$\ d\ = \sqrt{16 + 4 + 1} = \sqrt{21} \approx 4.583$	Needed to turn d into a unit vector.
Component $= \frac{F \cdot d}{\ d\ } = \frac{9}{\sqrt{21}} \approx 1.964$	The dot product with the unit vector $d/\ d\ $.
Projection $= \frac{F \cdot d}{d \cdot d} d = \frac{9}{21}(4, 2, 1) = \frac{3}{7}(4, 2, 1)$	The component times the unit vector, which divides by $\ d\ $ twice.

$W = 9$, **component** $\frac{9}{\sqrt{21}} \approx 1.964$,
projection $(\frac{12}{7}, \frac{6}{7}, \frac{3}{7})$

Check: F minus the projection is
 $(\frac{2}{7}, -\frac{13}{7}, \frac{18}{7})$, and its dot product with d is
 $\frac{8-26+18}{7} = 0$.

Problem 3

For $u = (1, 2, 2)$ and $v = (3, 0, 4)$, find $u \times v$, check that it is perpendicular to both, and find the area of the parallelogram and of the triangle on u and v .

First entry: $u_2v_3 - u_3v_2 = 2 \cdot 4 - 2 \cdot 0 = 8$

Cover the first column and take the 2×2 determinant of what is left.

Second entry: $u_3v_1 - u_1v_3 = 2 \cdot 3 - 1 \cdot 4 = 2$; third: $u_1v_2 - u_2v_1 = 0 - 6 = -6$

The middle entry has the order reversed, which is the minus sign of the cofactor expansion (Mistake 1).

$(8, 2, -6) \cdot (1, 2, 2) = 8 + 4 - 12 = 0$ and
 $(8, 2, -6) \cdot (3, 0, 4) = 24 + 0 - 24 = 0$

The check catches almost every slip.

$\|u \times v\| = \sqrt{64 + 4 + 36} = \sqrt{104} = 2\sqrt{26} \approx 10.198$

Check with Lagrange's identity:
 $\|u\|^2\|v\|^2 - (u \cdot v)^2 = 225 - 121 = 104$.

$u \times v = (8, 2, -6)$; **parallelogram area**
 $2\sqrt{26} \approx 10.20$, **triangle area** $\sqrt{26} \approx 5.10$

The triangle is half the parallelogram.

Problem 4

Find the area of the triangle with vertices $A(1, 0, 0)$, $B(0, 2, 0)$, $C(0, 0, 3)$, and the equation of the plane through them.

$\vec{AB} = (-1, 2, 0)$ and $\vec{AC} = (-1, 0, 3)$

Two edges from the same vertex.

$n = \vec{AB} \times \vec{AC} = (2 \cdot 3 - 0 \cdot 0, 0 \cdot (-1) - (-1) \cdot 3, (-1) \cdot 0 - 2 \cdot (-1)) = (6, 3, 2)$

Perpendicular to both edges, so normal to the plane.

Area $= \frac{1}{2}\|n\| = \frac{1}{2}\sqrt{36 + 9 + 4} = \frac{7}{2}$

Half the parallelogram on the two edges.

$6x + 3y + 2z = n \cdot A = 6$	The constant comes from a point on the plane. Check B : 6; C : 6.
Area $\frac{7}{2}$; plane $6x + 3y + 2z = 6$	Dividing by 6 gives the intercept form $\frac{x}{1} + \frac{y}{2} + \frac{z}{3} = 1$. Taking the constant from the vector $\vec{AB}$ instead of a point gives a parallel plane through the origin (Mistake 4).

Problem 5

Find the volume of the parallelepiped on $a = (1, 0, 1)$, $b = (2, 1, 0)$, $c = (0, 1, 3)$, and the volume of the tetrahedron with the same three edges from one vertex.

$\det \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = 1(1 \cdot 3 - 0 \cdot 1) - 0(2 \cdot 3 - 0 \cdot 0) + 1(2 \cdot 1 - 1 \cdot 0)$	Expand along the first row.
$= 3 - 0 + 2 = 5$	The triple product $a \cdot (b \times c)$.
Check: $b \times c = (3, -6, 2)$ and $a \cdot (3, -6, 2) = 3 + 0 + 2 = 5$	The same number computed the other way.
The tetrahedron is $\frac{1}{6}$ of the parallelepiped	Half for the triangular base, a third for the pyramid.
Parallelepiped 5, tetrahedron $\frac{5}{6}$	The sign of the determinant is positive, so a, b, c are a right-handed triple. The product of the lengths, $\sqrt{2} \cdot \sqrt{5} \cdot \sqrt{10} = 10$, is the volume only when the edges are perpendicular (Mistake 5).

Problem 6

Find the line through $P(1, 2, 3)$ and $Q(4, 0, 5)$, and the point where it meets the plane $x + y + z = 12$.

$d = Q - P = (3, -2, 2)$	The direction is the difference of two points on the line.
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$r(t) = (1 + 3t, 2 - 2t, 3 + 2t)$	$t = 0$ gives P and $t = 1$ gives Q .
$(1 + 3t) + (2 - 2t) + (3 + 2t) = 6 + 3t = 12$	Substitute the line into the plane: one equation in t .
$t = 2$	So the meeting point is beyond Q .
$r(t) = (1, 2, 3) + t(3, -2, 2)$; it meets the plane at $(7, -2, 7)$	Check: $7 - 2 + 7 = 12$. If $n \cdot d$ had been 0 the line would be parallel to the plane, and the equation in step 3 would have no t in it.

Problem 7

Find the distance from $Q(2, 1, 3)$ to the plane $2x - y + 2z = 4$, and the point of the plane nearest to Q .

$n = (2, -1, 2)$ and $\ n\ = 3$	The normal is read off the coefficients.
$n \cdot Q - 4 = 4 - 1 + 6 - 4 = 5$	How far the equation is from being satisfied, in units of $\ n\ $.
Distance $= \frac{ 5 }{3} = \frac{5}{3}$	Dividing by $\ n\ $ turns it into a length.
Nearest point: $Q - \frac{5}{\ n\ ^2} n = (2, 1, 3) - \frac{5}{9}(2, -1, 2) = \left(\frac{8}{9}, \frac{14}{9}, \frac{17}{9}\right)$	Move from Q along the normal by the right amount. Check: $\frac{16-14+34}{9} = 4$.
Distance $\frac{5}{3}$; nearest point $\left(\frac{8}{9}, \frac{14}{9}, \frac{17}{9}\right)$	The distance from Q to that point is $\frac{5}{9}\ n\ = \frac{5}{3}$. Forgetting to divide by $\ n\ $ gives 5 (Mistake 3).

Problem 8

Find the angle between the planes $x + y + z = 1$ and $x - y + 2z = 3$, and their line of intersection.

$$n_1 = (1, 1, 1), n_2 = (1, -1, 2), n_1 \cdot n_2 = 1 - 1 + 2 = 2$$

The angle between two planes is the angle between their normals.

$$\cos \theta = \frac{2}{\sqrt{3}\sqrt{6}} = \frac{\sqrt{2}}{3} \approx 0.4714, \text{ so } \theta \approx 61.87^\circ$$

$$\sqrt{18} = 3\sqrt{2}.$$

$$d = n_1 \times n_2 = (1 \cdot 2 - 1 \cdot (-1), 1 \cdot 1 - 1 \cdot 2, 1 \cdot (-1) - 1 \cdot 1) = (3, -1, -2)$$

The line lies in both planes, so it is perpendicular to both normals.

$$\text{With } z = 0: x + y = 1 \text{ and } x - y = 3, \text{ so } (x, y) = (2, -1)$$

Any point on both planes will do.

$$\theta \approx 61.87^\circ; \text{ line } (2, -1, 0) + t(3, -1, -2)$$

Check a second point, $t = 1$: $(5, -2, -2)$ gives $5 - 2 - 2 = 1$ and $5 + 2 - 4 = 3$.

Problem 9

Show that the lines $L_1 : (1, 0, 0) + s(1, 1, 0)$ and $L_2 : (0, 0, 2) + t(0, 1, 1)$ do not meet and are not parallel, and find the distance between them.

The directions $(1, 1, 0)$ and $(0, 1, 1)$ are not multiples of each other

So the lines are not parallel.

Meeting would need $1 + s = 0, s = t, 0 = 2 + t$: $s = -1$ from the first, $t = -2$ from the third, and then $s \neq t$

The three equations have no common solution: the lines are skew.

$$n = (1, 1, 0) \times (0, 1, 1) = (1, -1, 1)$$

Perpendicular to both lines.

$P_2 - P_1 = (-1, 0, 2)$ and $(P_2 - P_1) \cdot n = -1 + 0 + 2 = 1$	Project the vector between any two points of the lines onto the common normal.
Distance $\frac{1}{\sqrt{3}} \approx 0.5774$	Divide by $\ n\ = \sqrt{3}$. The closest points are at $s = -\frac{4}{3}$ and $t = -\frac{5}{3}$: $(-\frac{1}{3}, -\frac{4}{3}, 0)$ on L_1 and $(0, -\frac{5}{3}, \frac{1}{3})$ on L_2 . Their difference is $\frac{1}{3}(1, -1, 1)$, a multiple of n , of length $\frac{1}{\sqrt{3}}$.

Problem 10

A linear classifier predicts the positive class when $w \cdot x + b > 0$, with $w = (2, -1, 2)$ and $b = -3$. Find the signed distances of $x_1 = (2, 0, 1)$ and $x_2 = (0, 1, 0)$ to the decision boundary, and the point of the boundary nearest to x_1 .

The boundary is the plane $w \cdot x = 3$, with normal w and $\ w\ = 3$	The weight vector is perpendicular to the decision boundary.
$w \cdot x_1 + b = 4 + 0 + 2 - 3 = 3$, so the signed distance is $\frac{3}{3} = 1$	Positive: x_1 is on the positive side.
$w \cdot x_2 + b = 0 - 1 + 0 - 3 = -4$, so the signed distance is $-\frac{4}{3}$	Negative: x_2 is classified negative, $\frac{4}{3}$ from the boundary.
Nearest point to x_1 : $x_1 - 1 \cdot \frac{w}{\ w\ } = (2, 0, 1) - \frac{1}{3}(2, -1, 2) = (\frac{4}{3}, \frac{1}{3}, \frac{1}{3})$	Step back along the unit normal by the signed distance. Check: $\frac{8}{3} - \frac{1}{3} + \frac{2}{3} - 3 = 0$.
Signed distances 1 and $-\frac{4}{3}$; nearest boundary point $(\frac{4}{3}, \frac{1}{3}, \frac{1}{3})$	The raw score $w \cdot x + b$ is the signed distance times $\ w\ $, so scaling w and b by 10 multiplies every score by 10 without moving the boundary. That is why a margin is measured as $\frac{w \cdot x + b}{\ w\ }$.

Where this goes wrong

1. The middle sign of the cross product

The three entries are computed with the same pattern, $u_i v_j - u_j v_i$ in index order.

$u = (1, 2, 2), v = (3, 0, 4)$, first entry $u_2 v_3 - u_3 v_2 = 8$	Right so far: Problem 3.
“Second entry: $u_1 v_3 - u_3 v_1 = 4 - 6 = -2$. ”	The step that causes the mistake: the middle entry is $u_3 v_1 - u_1 v_3$, the minus sign of the cofactor expansion.
$u \times v = (8, -2, -6) \times$	The check catches it: $(8, -2, -6) \cdot (1, 2, 2) = 8 - 4 - 12 = -8 \neq 0$, so this vector is not perpendicular to u . The cross product is $(8, 2, -6)$.

2. Dividing by the squared lengths

The angle formula is remembered with $u \cdot u$ and $v \cdot v$ in the denominator.

$u \cdot v = 11, \ u\ ^2 = 9, \ v\ ^2 = 25$	Right so far: Problem 1.
“ $\cos \theta = \frac{u \cdot v}{\ u\ ^2 \ v\ ^2}$.”	The squared lengths belong to the projection coefficient $\frac{u \cdot v}{u \cdot u}$, not to the cosine.
$\cos \theta = \frac{11}{225} \approx 0.0489$, so $\theta \approx 87.2^\circ \times$	Nearly perpendicular, for two vectors whose dot product is large and positive. A cosine must not change when u is doubled; this one halves. With $\ u\ \ v\ = 15, \theta \approx 42.83^\circ$.

3. The distance without dividing by $|n|$

The value of the plane's equation at the point is taken as the distance.

$n \cdot Q - 4 = 5$ for $Q = (2, 1, 3)$ and the plane $2x - y + 2z = 4$	Right so far: Problem 7.
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“The distance is $ n \cdot Q - 4 $.”	That number depends on how the equation is written: $4x - 2y + 4z = 8$ is the same plane and would give 10.
Distance = 5 ✗	5 is the distance measured in units of $\ n\ = 3$. Divide: the distance is $\frac{5}{3}$. Check with the nearest point $(\frac{8}{9}, \frac{14}{9}, \frac{17}{9})$, which is $\frac{5}{3}$ from Q .

4. The plane through the wrong point

The constant on the right of the plane's equation is computed from an edge vector.

$n = (6, 3, 2)$ is normal to the plane through A, B, C	Right so far: Problem 4.
“ $k = n \cdot \vec{AB} = -6 + 6 + 0 = 0$.”	$\vec{AB}$ lies in the plane's direction but is not a point of it; n is perpendicular to every edge, so this always gives 0.
The plane is $6x + 3y + 2z = 0$ ✗	That plane passes through the origin, parallel to the right one. A does not satisfy it: $6 \neq 0$. Use a point: $k = n \cdot A = 6$.

5. The product of the lengths as a volume

The volume of a box is length times width times height, and the same is done for a slanted one.

$\ a\ = \sqrt{2}, \ b\ = \sqrt{5}, \ c\ = \sqrt{10}$	Right so far: the edges of Problem 5.
“Volume = $\ a\ \ b\ \ c\ $.”	True only when the edges are perpendicular; here $a \cdot b = 2$ and $a \cdot c = 3$.
Volume = $\sqrt{2 \cdot 5 \cdot 10} = 10$ ✗	Twice the true volume. The product of the lengths is an upper bound (Hadamard's inequality), reached only by a rectangular box. The triple product gives 5.

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