Surface integrals and the divergence theorem: dS, flux and Stokes

Ten problems on surface integrals: the area of a sphere from |r_φ × r_θ|, the area of a paraboloid as a graph, the mass of a hemispherical shell, the upward flux through a paraboloid cap with the normal (−g_x, −g_y, 1), the outward flux through a sphere, the divergence theorem on a box and on a ball, the flux through a hemisphere found by closing it with a disk, Stokes' theorem on a hemisphere and its boundary circle, and Gauss's law for the field r/|r|³ with the origin inside and outside, with worked solutions and the mistakes of dropping |r_u × r_v|, the normal (g_x, g_y, 1), forgetting the disk that closes the surface, a boundary traced the wrong way for Stokes and the divergence theorem across a singularity.

BEFORE YOU START

A surface integral adds something up over a curved surface the way a line integral adds it up along a curve. With respect to area it totals a density over a shell; with a vector field it totals the flux, the rate at which the field flows through the surface. Each one is computed the same way: parametrize the surface by two variables, find the area element $dS = \|r_u \times r_v\|\,du\,dv$ from a cross product, and do an ordinary double integral. Two theorems save most of the work. The divergence theorem trades the flux out of a closed surface for a triple integral of the divergence over the solid inside:

$$\iint_S F \cdot n\,dS = \iiint_E \nabla \cdot F\,dV,$$

and Stokes' theorem trades the flux of a curl for a line integral around the boundary. They are Green's theorem in three dimensions, and they are where Gauss's law of electrostatics comes from. These ten problems compute areas, masses and fluxes directly, then with each theorem, and end with the point charge whose field the divergence theorem gets wrong if you let it.

- A surface S is given by $r(u, v) = (x, y, z)$ over a region D of the uv-plane. The vectors r_u and r_v are tangent to it, $r_u \times r_v$ is normal to it, and its area element is $dS = \|r_u \times r_v\|\,du\,dv$.
- **Graphs.** For $z = g(x, y)$ over D, take $r = (x, y, g)$: then $r_x \times r_y = (-g_x, -g_y, 1)$, which points up, and $dS = \sqrt{1 + g_x^2 + g_y^2}\,dA$.
- **Spheres.** On the sphere of radius a, $r = (a\sin\phi\cos\theta,\ a\sin\phi\sin\theta,\ a\cos\phi)$ with $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$. The outward unit normal is $n = r/a$ and $dS = a^2\sin\phi\,d\phi\,d\theta$.
- **Scalar surface integral.** $\iint_S g\,dS = \iint_D g(r(u,v))\,\|r_u \times r_v\|\,du\,dv$. It does not depend on which way the normal points.

- **Flux.** $\iint_S F \cdot n\, dS = \iint_D F(r(u,v)) \cdot (r_u \times r_v)\, du\, dv$, with n the chosen unit normal; if $r_u \times r_v$ points the other way, flip its sign. For a closed surface the convention is the outward normal.
- **Divergence and curl.** For $F = (P, Q, R)$: $\nabla \cdot F = P_x + Q_y + R_z$, and $\nabla \times F = (R_y - Q_z,\ P_z - R_x,\ Q_x - P_y)$.
- **Divergence theorem.** If S is the whole boundary of the solid E, with the outward normal, and F has continuous partial derivatives on all of E: $\iint_S F \cdot n\, dS = \iiint_E \nabla \cdot F\, dV$.
- **Stokes' theorem.** If S has boundary curve C: $\iint_S (\nabla \times F) \cdot n\, dS = \oint_C F \cdot dr$, with C traced so that the surface is on your left when you walk along it with your head pointing along n (the right-hand rule).

Builds on: Line integrals and Green's theorem: work, potentials and circulation, Triple integrals: iterated limits, cylindrical and spherical coordinates, Vectors in 3-D: dot and cross products, lines and planes

Problems

1. Parametrize the sphere of radius 2 by ϕ and θ, compute $r_\phi \times r_\theta$ and its length, and use it to find the area of the sphere. difficulty ·

2. Find the area of the part of the paraboloid $z = x^2 + y^2$ below the plane $z = 2$. difficulty ·

3. A thin shell has the shape of the upper hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$, and its density is $\rho = z$. Find its mass, and the average height of a point on the shell. difficulty ··

4. Find the upward flux of $F = (x, y, z)$ through the cap $z = 1 - x^2 - y^2$, $z \geq 0$. difficulty ··

5. Find the outward flux of $F = (0, 0, z)$ through the sphere of radius 2 about the origin, directly from the parametrization. difficulty ··

6. Use the divergence theorem to find the outward flux of $F = (x^2, y^2, z^2)$ out of the box $0 \leq x \leq 1$, $0 \leq y \leq 2$, $0 \leq z \leq 3$. Check it face by face. difficulty ·

7. Find the outward flux of $F = (x^3, y^3, z^3)$ through the unit sphere with the divergence theorem, and check it with the surface integral. difficulty ··

8. Let $F = (x + y^2,\ y + z^2,\ z + x^2)$ and let S be the upper hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$, with the upward normal. Find the flux of F through S by closing the surface with a disk. difficulty ···

9. Let $F = (z - y,\ x + z^2,\ xy)$ and let S be the upper unit hemisphere with the upward normal. Verify Stokes' theorem: compute $\iint_S (\nabla \times F) \cdot n \, dS$ and the line integral around the boundary circle. difficulty ···

10. A point charge at the origin has electric field proportional to difficulty ···

$$F = \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}}.$$

Show that $\nabla \cdot F = 0$ away from the origin. Find the outward flux of F through the sphere of radius a about the origin, through the surface of the cube $-1 \leq x, y, z \leq 1$, and through the unit sphere centered at $(3, 0, 0)$.

Answers

1. Area $= \int_0^{2\pi} \int_0^{\pi} 4 \sin\phi \, d\phi \, d\theta = 4 \cdot 2 \cdot 2\pi = 16\pi$
2. Area $= 2\pi \cdot \frac{13}{6} = \frac{13\pi}{3} \approx 13.61$
3. Mass 8π; the average height is $8\pi / 8\pi = 1$
4. Flux $= \frac{3\pi}{2} \approx 4.71$
5. Flux $= 8 \cdot \frac{2}{3} \cdot 2\pi = \frac{32\pi}{3} \approx 33.51$
6. Flux $= 36$; the faces give $6 + 12 + 18 = 36$
7. Flux $= \frac{12\pi}{5} \approx 7.54$ both ways
8. Flux through $S = 2\pi - \left(-\frac{\pi}{4}\right) = \frac{9\pi}{4} \approx 7.07$
9. Both sides equal 2π
10. Flux 4π through every sphere about the origin, 4π through the cube, and 0 through the sphere centered at $(3, 0, 0)$

Worked solutions

Problem 1

> Parametrize the sphere of radius 2 by ϕ and θ, compute $r_\phi \times r_\theta$ and its length, and use it to find the area of the sphere.

$r = (2\sin\phi\cos\theta,\ 2\sin\phi\sin\theta,\ 2\cos\phi)$	ϕ is measured down from the positive z-axis, θ around it.
$r_\phi = (2\cos\phi\cos\theta,\ 2\cos\phi\sin\theta,\ -2\sin\phi)$, $r_\theta = (-2\sin\phi\sin\theta,\ 2\sin\phi\cos\theta,\ 0)$	Differentiate each coordinate.
$r_\phi \times r_\theta =$ $4\sin\phi\,(\sin\phi\cos\theta,\ \sin\phi\sin\theta,\ \cos\phi)$	The components are $4\sin^2\phi\cos\theta$, $4\sin^2\phi\sin\theta$ and $4\sin\phi\cos\phi$. The vector in parentheses is $r/2$, so the cross product points outward.
$\lVert r_\phi \times r_\theta \rVert = 4\sin\phi$	The vector in parentheses is a unit vector, and $\sin\phi \geq 0$ on $[0, \pi]$.
Area $= \int_0^{2\pi}\int_0^{\pi} 4\sin\phi\,d\phi\,d\theta = 4\cdot 2\cdot 2\pi =$ 16π	It agrees with $4\pi a^2$. Integrating $d\phi\,d\theta$ with no factor gives $2\pi^2 \approx 19.74$ instead of $16\pi \approx 50.27$: the rectangle of angles is not the sphere (Mistake 1).

Problem 2

> Find the area of the part of the paraboloid $z = x^2 + y^2$ below the plane $z = 2$.

$g_x = 2x$, $g_y = 2y$, so $dS =$ $\sqrt{1 + 4x^2 + 4y^2}\,dA$	The graph formula.
The shadow is the disk $x^2 + y^2 \leq 2$	The paraboloid meets $z = 2$ where $r^2 = 2$.
Area $= \int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{1 + 4r^2}\,r\,dr\,d\theta$	Polar coordinates, keeping the r of $dA = r\,dr\,d\theta$.

$\int_0^{\sqrt{2}} \sqrt{1+4r^2}\, r\, dr = \frac{1}{8}\int_1^9 \sqrt{u}\, du = \frac{1}{12}(27-1) = \frac{13}{6}$	Substitute $u = 1+4r^2$, $du = 8r\, dr$.
Area $= 2\pi \cdot \frac{13}{6} = \frac{13\pi}{3} \approx 13.61$	More than twice the area $2\pi \approx 6.28$ of the disk underneath, because the surface is steep near the rim. Leaving out the square root gives exactly the disk's area (Mistake 1).

Problem 3

> A thin shell has the shape of the upper hemisphere $x^2+y^2+z^2 = 4$, $z \geq 0$, and its density is $\rho = z$. Find its mass, and the average height of a point on the shell.

$z = 2\cos\phi$ and $dS = 4\sin\phi\, d\phi\, d\theta$, with $0 \leq \phi \leq \frac{\pi}{2}$	Problem 1 with the upper half only.
$\text{Mass} = \int_0^{2\pi}\int_0^{\pi/2} 2\cos\phi \cdot 4\sin\phi\, d\phi\, d\theta$	A scalar surface integral; the direction of the normal plays no part.
$\int_0^{\pi/2} \cos\phi \sin\phi\, d\phi = \frac{1}{2}$	It is $\frac{1}{2}\sin^2\phi$ from 0 to $\frac{\pi}{2}$.
$\text{Mass} = 8 \cdot \frac{1}{2} \cdot 2\pi = 8\pi \approx 25.13$	The θ-integral gives 2π.
Mass 8π**; the average height is** $8\pi/8\pi = 1$	The area of the hemisphere is $2\pi a^2 = 8\pi$. The average height of the shell is $\frac{a}{2}$, higher than the $\frac{3a}{8}$ of the solid half-ball, whose volume crowds near the flat base.

Problem 4

> Find the upward flux of $F = (x, y, z)$ through the cap $z = 1 - x^2 - y^2$, $z \geq 0$.

$g = 1 - x^2 - y^2$, so $(-g_x, -g_y, 1) = (2x, 2y, 1)$	It points up, as required: the z-component is 1.

$F \cdot (2x, 2y, 1) = 2x^2 + 2y^2 + (1 - x^2 - y^2) = 1 + r^2$	On the surface, $z = 1 - r^2$.
The shadow is the unit disk	The cap meets $z = 0$ where $r = 1$.
$\text{Flux} = \int_0^{2\pi} \int_0^1 (1 + r^2)\, r\, dr\, d\theta = 2\pi\left(\frac{1}{2} + \frac{1}{4}\right)$	The $\lVert r_x \times r_y \rVert$ cancels against the $1/\lVert r_x \times r_y \rVert$ in the unit normal, so no square root appears.
Flux $= \frac{3\pi}{2} \approx 4.71$	A check with the divergence theorem: $\nabla \cdot F = 3$ and the solid under the cap has volume $\frac{\pi}{2}$, giving $\frac{3\pi}{2}$ out of the closed surface; on the flat bottom $F \cdot (0, 0, -1) = -z = 0$, so all of it leaves through the cap. With $(g_x, g_y, 1)$ as the normal the answer is $-\frac{\pi}{2}$ (Mistake 2).

Problem 5

> Find the outward flux of $F = (0, 0, z)$ through the sphere of radius 2 about the origin, directly from the parametrization.

$n = \frac{1}{2}(x, y, z)$, so $F \cdot n = \frac{z^2}{2}$	The outward unit normal of a sphere is the position vector divided by the radius.
$F \cdot n = 2\cos^2\phi$ and $dS = 4\sin\phi\, d\phi\, d\theta$	$z = 2\cos\phi$.
$\text{Flux} = \int_0^{2\pi} \int_0^{\pi} 8\cos^2\phi \sin\phi\, d\phi\, d\theta$	Both hemispheres add: below the equator $z < 0$ and the field points down, which is still outward.
$\int_0^{\pi} \cos^2\phi \sin\phi\, d\phi = \frac{2}{3}$	It is $-\frac{1}{3}\cos^3\phi$ from 0 to π.
Flux $= 8 \cdot \frac{2}{3} \cdot 2\pi = \frac{32\pi}{3} \approx 33.51$	This is the volume of the ball, $\frac{4}{3}\pi \cdot 2^3$, as the divergence theorem predicts: $\nabla \cdot F = 1$.

Problem 6

> Use the divergence theorem to find the outward flux of $F = (x^2, y^2, z^2)$ out of the box $0 \le x \le 1$, $0 \le y \le 2$, $0 \le z \le 3$. Check it face by face.

$\nabla \cdot F = 2x + 2y + 2z$	Sum of P_x, Q_y, R_z.
$\iiint 2x\,dV = 2 \cdot \frac{1}{2} \cdot 6 = 6$, $\iiint 2y\,dV = 2 \cdot 1 \cdot 6 = 12$, $\iiint 2z\,dV = 2 \cdot \frac{3}{2} \cdot 6 = 18$	The box has volume 6, and the average of x, y, z over it is the center $(\frac{1}{2}, 1, \frac{3}{2})$.
Total $= 6 + 12 + 18 = 36$	The divergence theorem's answer.
On $x = 1$: $F \cdot n = 1$ over area 6; on $y = 2$: 4 over area 3; on $z = 3$: 9 over area 2	The faces $x = 0$, $y = 0$, $z = 0$ contribute 0, because there the normal component of F is $-x^2 = 0$, $-y^2 = 0$, $-z^2 = 0$.
Flux $= 36$; **the faces give** $6 + 12 + 18 = 36$	Each face's flux matches one term of the triple integral.

Problem 7

> Find the outward flux of $F = (x^3, y^3, z^3)$ through the unit sphere with the divergence theorem, and check it with the surface integral.

$\nabla \cdot F = 3x^2 + 3y^2 + 3z^2 = 3\rho^2$	In spherical coordinates.
$\iiint 3\rho^2\,dV = \int_0^{2\pi} \int_0^{\pi} \int_0^1 3\rho^4 \sin\phi \, d\rho \, d\phi \, d\theta$	$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$.
$= 3 \cdot \frac{1}{5} \cdot 2 \cdot 2\pi = \frac{12\pi}{5}$	The limits are numbers, so the integral splits.
On the sphere $n = (x, y, z)$, so $F \cdot n = x^4 + y^4 + z^4$, and $\iint x^4\,dS = \frac{4\pi}{5}$	With $z = \cos\phi$: $\iint z^4\,dS = 2\pi \int_0^{\pi} \cos^4\phi \sin\phi \, d\phi = 2\pi \cdot \frac{2}{5}$. By symmetry x^4 and y^4 give the same.
Flux $= \frac{12\pi}{5} \approx 7.54$ **both ways**	$3 \cdot \frac{4\pi}{5}$ from the surface.

Problem 8

Let $F = (x + y^2,\ y + z^2,\ z + x^2)$ and let S be the upper hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$, with the upward normal. Find the flux of F through S by closing the surface with a disk.

S and the unit disk D in the plane $z = 0$ together bound the half-ball E	The divergence theorem needs a closed surface; S alone is not one.
$\nabla \cdot F = 1 + 1 + 1 = 3$, so $\iiint_E 3\,dV = 3 \cdot \frac{2\pi}{3} = 2\pi$	The half-ball has volume $\frac{2}{3}\pi$.
On D the outward normal is $(0, 0, -1)$ and $F \cdot n = -(z + x^2) = -x^2$	Outward from E is down on the base.
$\iint_D -x^2\,dA = -\int_0^{2\pi}\int_0^1 r^3\cos^2\theta\,dr\,d\theta = -\frac{\pi}{4}$	$\frac{1}{4} \cdot \pi$.
Flux through $S = 2\pi - \left(-\frac{\pi}{4}\right) = \frac{9\pi}{4} \approx 7.07$	The total out of E is 2π, and the base takes $-\frac{\pi}{4}$ of it, so the dome carries the rest. Stopping at 2π forgets the base (Mistake 3).

Problem 9

Let $F = (z - y,\ x + z^2,\ xy)$ and let S be the upper unit hemisphere with the upward normal. Verify Stokes' theorem: compute $\iint_S (\nabla \times F) \cdot n\,dS$ and the line integral around the boundary circle.

$\nabla \times F = (x - 2z,\ 1 - y,\ 2)$	$R_y - Q_z = x - 2z$, $P_z - R_x = 1 - y$, $Q_x - P_y = 1 + 1$.
With $n = (x, y, z)$: $(\nabla \times F) \cdot n = x^2 - 2xz + y - y^2 + 2z$	The upward normal of the upper hemisphere is outward.
$\iint_S x^2\,dS = \iint_S y^2\,dS = \frac{2\pi}{3}$, $\iint_S xz\,dS = \iint_S y\,dS = 0$, $\iint_S z\,dS = \pi$	Half of $\frac{4\pi}{3}$ for each square; the odd terms cancel between θ and $\theta + \pi$; and $\iint z\,dS$ is Problem 3 with radius 1.

$\iint_S (\nabla \times F) \cdot n \, dS = \frac{2\pi}{3} - \frac{2\pi}{3} + 2\pi = 2\pi$	Add the pieces.
Both sides equal 2π	Counterclockwise seen from above, $r = (\cos t, \sin t, 0)$, so $F = (-\sin t, \cos t, \sin t \cos t)$ and $F \cdot r' = \sin^2 t + \cos^2 t = 1$, giving 2π. The flat unit disk has the same boundary and gives $\iint 2 \, dA = 2\pi$ too. Clockwise gives -2π (Mistake 4).

Problem 10

A point charge at the origin has electric field proportional to

$$F = \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}}.$$

Show that $\nabla \cdot F = 0$ away from the origin. Find the outward flux of F through the sphere of radius a about the origin, through the surface of the cube $-1 \le x, y, z \le 1$, and through the unit sphere centered at $(3, 0, 0)$.

$P_x = \frac{1}{\rho^3} - \frac{3x^2}{\rho^5}$, and similarly for Q_y and R_z	Here $\rho = \sqrt{x^2 + y^2 + z^2}$, and $\partial \rho / \partial x = x/\rho$.
$\nabla \cdot F = \frac{3}{\rho^3} - \frac{3\rho^2}{\rho^5} = 0$ for $\rho \neq 0$	The three x^2, y^2, z^2 terms add up to ρ^2.
On the sphere of radius a: $F \cdot n = \frac{1}{a^2}$, so the flux is $\frac{1}{a^2} \cdot 4\pi a^2 = 4\pi$	F points straight out and has length $\frac{1}{a^2}$. The radius cancels.
Cube: apply the divergence theorem to the region between the cube and a small sphere inside it, where F is smooth: the flux out of the cube equals the flux out of the small sphere, 4π	The origin is not in that region, and $\nabla \cdot F = 0$ there. Each face of the cube carries $\frac{4\pi}{6} = \frac{2\pi}{3}$ by symmetry.

Flux 4π through every sphere about the origin, 4π through the cube, and 0 through the sphere centered at $(3, 0, 0)$	That sphere does not contain the origin, so the divergence theorem applies on its ball and gives $\iiint 0 \, dV = 0$. This is Gauss's law: the flux counts the charge inside, whatever the surface. Using $\nabla \cdot F = 0$ on a ball that contains the origin gives 0 instead of 4π (Mistake 5).

Where this goes wrong

1. Dropping |r_u × r_v| from dS

The surface integral is written as if dS were dA of the shadow, or $du\,dv$ of the parameters.

$z = x^2 + y^2$ below $z = 2$ has shadow $x^2 + y^2 \le 2$	Right so far: Problem 2.
"Area $= \iint_D 1 \, dA$."	The step that causes the mistake: $dS = \sqrt{1 + g_x^2 + g_y^2}\, dA$, and the tilt of the surface is what the square root measures.
Area $= 2\pi \approx 6.28$ ✗	That is the area of the flat disk, not of the bowl above it; the true area is $\frac{13\pi}{3} \approx 13.61$. The same slip on the sphere of Problem 1, integrating $d\phi\, d\theta$ with no $4 \sin\phi$, gives $2\pi^2 \approx 19.74$ for an area of $16\pi \approx 50.27$.

2. The normal (g_x, g_y, 1) instead of (−g_x, −g_y, 1)

The normal of a graph is remembered as the gradient of g with a 1 appended.

$g = 1 - x^2 - y^2$ and $F = (x, y, z)$	Right so far: Problem 4.
"The upward normal is $(g_x, g_y, 1) = (-2x, -2y, 1)$."	The surface is the level set $z - g = 0$, whose gradient is $(-g_x, -g_y, 1)$. The vector $(g_x, g_y, 1)$ is not normal to the surface at all.

$F \cdot n\, dS = (1 - 3r^2)\, dA$, so the flux is $2\pi\left(\frac{1}{2} - \frac{3}{4}\right) = -\frac{\pi}{2}$ ✗	The field points out of the cap everywhere, so the upward flux must be positive. With $(2x, 2y, 1)$ it is $\frac{3\pi}{2}$, which the divergence theorem confirms.

3. Forgetting the disk that closes the surface

The divergence theorem is applied to a hemisphere as if it were a closed surface.

$\nabla \cdot F = 3$ for $F = (x + y^2, y + z^2, z + x^2)$	Right so far: Problem 8.
"Flux through the hemisphere $= \iiint_E 3\, dV$."	The triple integral is the flux out of the whole boundary of E, which includes the flat disk at the bottom.
Flux through $S = 2\pi$ ✗	The disk carries $-\frac{\pi}{4}$ of that total, so the dome carries $2\pi + \frac{\pi}{4} = \frac{9\pi}{4} \approx 7.07$. The disk's flux is 0 only when $F_z = 0$ on the plane $z = 0$, as in Problem 4.

4. A boundary traced the wrong way for Stokes

The boundary circle is traced in whichever direction comes first.

$\iint_S (\nabla \times F) \cdot n\, dS = 2\pi$ with the upward normal	Right so far: Problem 9.
"$r = (\cos t, -\sin t, 0)$, $0 \le t \le 2\pi$."	This runs clockwise seen from above. With the normal pointing up, the right-hand rule needs counterclockwise.
$\oint_C F \cdot dr = -2\pi$, so Stokes' theorem seems to fail ✗	Clockwise, $F = (\sin t, \cos t, -\sin t \cos t)$ and $r' = (-\sin t, -\cos t, 0)$, so $F \cdot r' = -1$ at every point. Reversing the curve flips the sign; counterclockwise gives 2π, as the surface does.

5. The divergence theorem across a singularity

The divergence of the point-charge field is zero, so every closed flux is assumed to be zero.

$\nabla \cdot F = 0$ for $\rho \neq 0$	Right so far: Problem 10.
"By the divergence theorem the flux out of the unit sphere is $\iiint_{\rho \leq 1} 0 \, dV$."	The theorem needs F smooth on the whole solid, and F is undefined at the origin, which is inside the ball.
Flux $= 0$ ✗	The direct computation gives $F \cdot n = 1$ at every point of the unit sphere and a flux of $4\pi \approx 12.57$. All of the divergence sits at the one point where F blows up; the theorem gives 0 only for surfaces that leave the origin outside, like the sphere centered at $(3, 0, 0)$.

More at eulerpad.app. This is one of 82 free practice sets with worked solutions at eulerpad.app/practice, alongside guides to the topics behind them.

EulerPad, the app, reads your handwritten math on iPad, iPhone and Android and checks it line by line. It's free to start.

eulerpad.app/practice/surface-integrals-and-the-divergence-theorem