

# Series convergence tests: geometric, divergence, integral, comparison and alternating

Ten problems on infinite series: a geometric series that starts at  $n = 1$ , a telescoping sum, the divergence test and what it cannot say about  $1/\sqrt{n}$ , the integral test on  $1/(n \ln^2 n)$  with bounds on the sum, a direct comparison for  $(2 + \cos n)/n^2$ , a limit comparison, the alternating harmonic series and conditional convergence, how many terms an alternating error bound needs, the ratio test on  $n!/n^n$ , and the step-size conditions behind stochastic gradient descent, with worked solutions and the mistakes of starting a geometric sum at the wrong term, reading terms that go to 0 as convergence and comparing with a bigger divergent series.

## BEFORE YOU START

An infinite series adds up infinitely many numbers, and the first question is whether the total is a finite number at all. Taylor series, the geometric sums behind an exponential moving average, and the step-size conditions in proofs that stochastic gradient descent converges all depend on the answer. A handful of tests settle most series: compare with a geometric series or a  $p$ -series, compare with an integral, or use the alternating signs. The work goes wrong in a few places: a geometric sum started at the wrong term, terms that go to 0 read as proof of convergence, a comparison made in the direction that proves nothing, the alternating series test used on terms that do not shrink, and a ratio limit of 1 treated as a verdict. These ten problems cover each of them.

- **Convergence.**  $\sum_{n=1}^{\infty} a_n$  converges to  $S$  if the partial sums  $S_N = a_1 + \dots + a_N$  tend to  $S$ . Otherwise it diverges.
- **Geometric series.**  $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$  for  $|r| < 1$ , and it diverges for  $|r| \geq 1$ . In general the sum is  $\frac{\text{first term}}{1-r}$ .
- **Divergence test.** If  $a_n \not\rightarrow 0$ , the series diverges. If  $a_n \rightarrow 0$ , this test says nothing.
- **Integral test and  $p$ -series.** If  $f$  is positive and decreasing with  $f(n) = a_n$ , then  $\sum a_n$  and  $\int_1^{\infty} f(x) dx$  converge or diverge together, and  $\int_N^{\infty} f \leq \sum_{n=N}^{\infty} a_n \leq a_N + \int_N^{\infty} f$ .  
So  $\sum \frac{1}{n^p}$  converges exactly when  $p > 1$ .
- **Comparison.** For  $0 \leq a_n \leq b_n$ : if  $\sum b_n$  converges so does  $\sum a_n$ ; if  $\sum a_n$  diverges so does  $\sum b_n$ . **Limit comparison:** if  $a_n, b_n > 0$  and  $a_n/b_n \rightarrow L$  with  $0 < L < \infty$ , both converge or both diverge.

- **Alternating series.** If  $b_n$  decreases to 0,  $\sum (-1)^{n+1} b_n$  converges, and stopping after  $N$  terms leaves an error of at most  $b_{N+1}$ . A series is **absolutely** convergent if  $\sum |a_n|$  converges, and **conditionally** convergent if it converges but  $\sum |a_n|$  does not.
- **Ratio test.** If  $|a_{n+1}/a_n| \rightarrow \ell$ , the series converges absolutely for  $\ell < 1$  and diverges for  $\ell > 1$ . At  $\ell = 1$  it says nothing.

Builds on: Improper integrals: infinite limits, infinite integrands and comparison,  
Evaluating limits: factoring, conjugates and limits at infinity

## Problems

1. Find  $\sum_{n=1}^{\infty} 5 \left(\frac{2}{3}\right)^n$ . difficulty ·

2. Find  $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$  from its partial sums. difficulty ·

3. What does the divergence test say about  $\sum_{n=1}^{\infty} \frac{n}{2n+1}$ , and about  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ ? difficulty ·

4. Use the integral test to show that  $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$  converges, and bound its sum. difficulty ··

5. Show that  $\sum_{n=1}^{\infty} \frac{2 + \cos n}{n^2}$  converges, and give bounds for its sum. difficulty ··

6. Does  $\sum_{n=1}^{\infty} \frac{3n+1}{n^3+2}$  converge? difficulty ··

7. Show that  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \dots$  converges, decide whether it converges absolutely, and bound the error of  $S_4$ . difficulty ··

**8.** How many terms of  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$  guarantee an error less than 0.001? difficulty ..

**9.** Use the ratio test on  $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ . difficulty ...

**10.** Proofs that stochastic gradient descent converges usually ask for step sizes  $\alpha_n > 0$  with  $\sum \alpha_n = \infty$  and  $\sum \alpha_n^2 < \infty$ . For which  $p$  does  $\alpha_n = \frac{c}{n^p}$  (with  $c > 0$ ) qualify? Does  $\alpha_n = 0.1 \cdot 0.99^n$ ? difficulty ...

## Answers

1. 10
2. 1
3. The first diverges; for the second the test is silent, and it diverges anyway
4. It converges, with  $1.443 \leq S \leq 2.483$
5. It converges, to a sum between 1.645 and 4.935
6. Yes, it converges
7. It converges conditionally, not absolutely;  $|S - S_4| \leq 0.2$
8. 10 terms
9.  $\ell = \frac{1}{e} \approx 0.368 < 1$ , so the series converges
10.  $\frac{1}{2} < p \leq 1$ ; the geometric schedule fails, because  $\sum \alpha_n = 9.9$  is finite

## Worked solutions

### Problem 1

Find  $\sum_{n=1}^{\infty} 5\left(\frac{2}{3}\right)^n$ .

$r = \frac{2}{3}$ , and  $|r| < 1$ , so the series converges

A geometric series: each term is  $\frac{2}{3}$  of the one before.

The first term, at  $n = 1$ , is  $5 \cdot \frac{2}{3} = \frac{10}{3}$

The sum starts at  $n = 1$ , not  $n = 0$ .

$$\frac{10/3}{1 - 2/3} = \frac{10/3}{1/3}$$

First term over  $1 - r$ .

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The partial sums  $\frac{10}{3}, \frac{50}{9}, \frac{190}{27}, \dots$  climb toward 10.

### Problem 2

Find  $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$  from its partial sums.

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

Partial fractions.

$$S_N = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{N} - \frac{1}{N+1}\right) = 1 - \frac{1}{N+1}$$

Every middle term cancels with the next one: the sum telescopes.

$$1 - \frac{1}{N+1} \rightarrow 1$$

Take the limit of the partial sums.

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For example  $S_{99} = 0.99$ . Telescoping sums are one of the few kinds, with geometric series, whose exact value is easy to find.

### Problem 3

What does the divergence test say about  $\sum_{n=1}^{\infty} \frac{n}{2n+1}$ , and about  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ ?

$$\frac{n}{2n+1} \rightarrow \frac{1}{2} \neq 0$$

Divide top and bottom by  $n$ .

So  $\sum \frac{n}{2n+1}$  diverges

Adding terms near  $\frac{1}{2}$  forever gives partial sums that grow like  $\frac{N}{2}$ .

$\frac{1}{\sqrt{n}} \rightarrow 0$ , so the divergence test says nothing about  $\sum \frac{1}{\sqrt{n}}$

Terms going to 0 are needed for convergence, but they are not enough.

**The first diverges; for the second the test is silent, and it diverges anyway**

It is a  $p$ -series with  $p = \frac{1}{2} \leq 1$ . Directly: each of the first  $N$  terms is at least  $\frac{1}{\sqrt{N}}$ , so  $S_N \geq \sqrt{N}$ .  $S_{100} \approx 18.59$ .

### Problem 4

Use the integral test to show that  $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$  converges, and bound its sum.

$f(x) = \frac{1}{x(\ln x)^2}$  is positive and decreasing for  $x \geq 2$

Both  $x$  and  $\ln x$  increase, so the denominator does.

$$\int_2^b \frac{dx}{x(\ln x)^2} = \left[ -\frac{1}{\ln x} \right]_2^b = \frac{1}{\ln 2} - \frac{1}{\ln b}$$

Substitute  $u = \ln x$ ,  $du = \frac{dx}{x}$ : the integral of  $u^{-2}$ .

$\frac{1}{\ln b} \rightarrow 0$ , so the integral is  $\frac{1}{\ln 2} \approx 1.4427$  and the series converges

The integral test.

$$\frac{1}{\ln 2} \leq S \leq \frac{1}{2(\ln 2)^2} + \frac{1}{\ln 2}$$

The bounds with  $N = 2$ : the integral alone, and the integral plus the first term.

**It converges, with**  $1.443 \leq S \leq 2.483$

The sum is about 2.110. Without the square,  $\sum \frac{1}{n \ln n}$  diverges: its integral is  $\ln(\ln b) \rightarrow \infty$ .

### Problem 5

Show that  $\sum_{n=1}^{\infty} \frac{2 + \cos n}{n^2}$  converges, and give bounds for its sum.

$$1 \leq 2 + \cos n \leq 3, \text{ so } \frac{1}{n^2} \leq \frac{2 + \cos n}{n^2} \leq \frac{3}{n^2}$$

$\cos n$  is between  $-1$  and  $1$ . Every term is positive.

$$\sum \frac{3}{n^2} \text{ converges}$$

A  $p$ -series with  $p = 2 > 1$ , times 3.

So the series converges by direct comparison

Smaller than a convergent series, term by term.

$$\sum \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.645 \text{ and } \sum \frac{3}{n^2} = \frac{\pi^2}{2} \approx 4.935$$

The known value of  $\sum \frac{1}{n^2}$ .

**It converges, to a sum between**  $1.645$  **and**  $4.935$

The sum is about 3.614. The integral test can't be used here, because the terms are not decreasing.

### Problem 6

Does  $\sum_{n=1}^{\infty} \frac{3n+1}{n^3+2}$  converge?

$$\text{For large } n, \frac{3n+1}{n^3+2} \text{ behaves like } \frac{3n}{n^3} = \frac{3}{n^2}$$

Keep the highest power on top and bottom.

$$\frac{(3n+1)/(n^3+2)}{1/n^2} = \frac{3n^3+n^2}{n^3+2} \rightarrow 3$$

Limit comparison with  $b_n = \frac{1}{n^2}$ .

$$0 < 3 < \infty \text{ and } \sum \frac{1}{n^2} \text{ converges}$$

Both series do the same thing.

## Yes, it converges

A direct comparison works too:  $3n + 1 \leq 4n$  and  $n^3 + 2 > n^3$ , so the terms are below  $\frac{4}{n^2}$ . The sum is about 3.260.

### Problem 7

Show that  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \dots$  converges, decide whether it converges absolutely, and bound the error of  $S_4$ .

$b_n = \frac{1}{n}$  decreases and tends to 0

The conditions of the alternating series test.

So the series converges

Each partial sum overshoots the last in the other direction, by less each time.

$\sum \left| \frac{(-1)^{n+1}}{n} \right| = \sum \frac{1}{n}$  diverges

The harmonic series, a  $p$ -series with  $p = 1$ .

$S_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12} \approx 0.5833$ , with error at most  $b_5 = \frac{1}{5}$

The first omitted term.

**It converges conditionally, not absolutely;**  $|S - S_4| \leq 0.2$

The sum is  $\ln 2 \approx 0.6931$ , so the actual error is about 0.110, inside the bound.

### Problem 8

How many terms of  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$  guarantee an error less than 0.001?

$b_n = \frac{1}{n^3}$  decreases to 0, so the error after  $N$  terms is at most  $\frac{1}{(N+1)^3}$

The alternating series error bound.

$\frac{1}{(N+1)^3} < 0.001$  needs  $(N+1)^3 > 1000$ , so  $N+1 > 10$

$10^3 = 1000$  exactly, which is not enough for a strict inequality.

|                                                          |                                                                                                                      |
|----------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------|
| $N = 10$ : the bound is $\frac{1}{1331} \approx 0.00075$ | $N = 9$ gives a bound of exactly 0.001.                                                                              |
| <b>10 terms</b>                                          | $S_{10} \approx 0.90112$ , and the sum is $\frac{3}{4}\zeta(3) \approx 0.90154$ : the actual error is about 0.00043. |

### Problem 9

|                                                                                       |                                                                                                                                                          |
|---------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------|
| Use the ratio test on $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ .                          |                                                                                                                                                          |
| $\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}$               | Write out the ratio of consecutive terms.                                                                                                                |
| $= \frac{(n+1)n^n}{(n+1)^{n+1}} = \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n$ | $(n+1)!/n! = n+1$ , which cancels one factor of $n+1$ .                                                                                                  |
| $\left(\frac{n}{n+1}\right)^n = \frac{1}{(1+1/n)^n} \rightarrow \frac{1}{e}$          | The limit that defines $e$ .                                                                                                                             |
| $\ell = \frac{1}{e} \approx 0.368 < 1$ , <b>so the series converges</b>               | The sum is about 1.880. Factorials and powers in the same term are what the ratio test is for: the comparison tests have nothing simple to compare with. |

### Problem 10

|                                                                                                                                                                                                                                                                        |                                                      |
|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------|
| Proofs that stochastic gradient descent converges usually ask for step sizes $\alpha_n > 0$ with $\sum \alpha_n = \infty$ and $\sum \alpha_n^2 < \infty$ . For which $p$ does $\alpha_n = \frac{c}{n^p}$ (with $c > 0$ ) qualify? Does $\alpha_n = 0.1 \cdot 0.99^n$ ? |                                                      |
| $\sum \frac{c}{n^p} = \infty$ exactly when $p \leq 1$                                                                                                                                                                                                                  | The $p$ -series test.                                |
| $\sum \left(\frac{c}{n^p}\right)^2 = c^2 \sum \frac{1}{n^{2p}} < \infty$ exactly when $2p > 1$                                                                                                                                                                         | Again the $p$ -series test, now with exponent $2p$ . |

|                                                                                                                    |                                                                                                                                                                                                                                          |
|--------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Both hold when $\frac{1}{2} < p \leq 1$                                                                            | So $\frac{c}{n}$ qualifies, and $\frac{c}{\sqrt{n}}$ (whose squares form the harmonic series) and $\frac{c}{n^2}$ (whose sum is finite) do not.                                                                                          |
| $\sum_{n=1}^{\infty} 0.1 \cdot 0.99^n = \frac{0.1 \cdot 0.99}{1 - 0.99} = 9.9$                                     | Geometric, with first term 0.099.                                                                                                                                                                                                        |
| $\frac{1}{2} < p \leq 1$ ; <b>the geometric schedule fails, because <math>\sum \alpha_n = 9.9</math> is finite</b> | The first condition lets the steps reach a minimum however far away it starts; with steps adding to 9.9, the iterates can only travel 9.9 times the largest gradient. The second condition makes the noise in the gradients average out. |

## Where this goes wrong

### 1. Starting the geometric sum at the wrong term

$\frac{a}{1-r}$  is memorized with  $a$  as the number in front of  $r^n$ .

|                                                                                             |                                                                                                                                                                      |
|---------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $\sum_{n=1}^{\infty} 5 \left(\frac{2}{3}\right)^n, \text{ geometric with } r = \frac{2}{3}$ | Right so far: Problem 1, step 1.                                                                                                                                     |
| “ $a = 5$ .”                                                                                | The habit that causes the mistake: 5 is the term at $n = 0$ , and this sum starts at $n = 1$ .                                                                       |
| $\frac{5}{1 - 2/3} = 15 \text{ ✗}$                                                          | That includes a term of 5 that is not in the series. The formula needs the first term actually in the sum, $5 \cdot \frac{2}{3} = \frac{10}{3}$ , and the sum is 10. |

### 2. Reading terms that go to 0 as convergence

The divergence test gets used backwards.

|                                    |                                  |
|------------------------------------|----------------------------------|
| $\frac{1}{\sqrt{n}} \rightarrow 0$ | Right so far: Problem 3, step 3. |
|------------------------------------|----------------------------------|

“The terms go to 0, so the series converges.”

The habit that causes the mistake: the divergence test turned around.

$$\sum \frac{1}{\sqrt{n}} \text{ converges } \times$$

It diverges:  $S_N \geq \sqrt{N}$ , and  $S_{10000} \approx 198.5$ . Terms that go to 0 are necessary for convergence, not sufficient, and the harmonic series  $\sum \frac{1}{n}$  is the standard example.

### 3. Comparing with a bigger divergent series

Any true inequality looks like a comparison.

$$a_n = \frac{3n+1}{n^3+2} > 0$$

Right so far: Problem 6.

“ $a_n \leq \frac{4}{n}$  for every  $n \geq 1$ , and  $\sum \frac{4}{n}$  diverges.”

The inequality is true:  $(3n+1)n \leq 4(n^3+2)$  for every  $n \geq 1$ .

So  $\sum a_n$  diverges  $\times$

Being smaller than a divergent series proves nothing; a series of zeros is smaller too. To prove divergence,  $a_n$  must be **at least** the terms of a divergent series; to prove convergence, at most those of a convergent one. Here  $a_n \leq \frac{4}{n^2}$ , and the series converges.

### 4. Using the alternating series test on terms that don't shrink to 0

The signs alternate, so the test seems to apply.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n+1}, \text{ with } b_n = \frac{n}{n+1}$$

The setup of Problem 7, on a different series.

“It alternates, so it converges.”

The habit that causes the mistake: the test has two conditions, and only the alternation was checked.

The series converges ✗

$b_n \rightarrow 1$ , not 0, so the terms do not go to 0 and the series diverges by the divergence test: its partial sums keep jumping by nearly 1. The alternating series test needs  $b_n$  to decrease **to** 0.

## 5. Taking a ratio limit of 1 as an answer

The ratio test is applied to every series, and its answer is read even at the boundary.

For  $\sum \frac{1}{n}$ :  $\frac{a_{n+1}}{a_n} = \frac{n}{n+1} \rightarrow 1$

Right so far: the same step as Problem 9, step 1.

“ $\ell \leq 1$ , so it converges.”

The habit that causes the mistake: the boundary case lumped in with  $\ell < 1$ .

$\sum \frac{1}{n}$  converges ✗

It diverges. At  $\ell = 1$  the ratio test gives no answer:  $\sum \frac{1}{n^2}$  also has ratio  $\left(\frac{n}{n+1}\right)^2 \rightarrow 1$  and converges. Any  $p$ -series has ratio limit 1, so for these, use the integral test or a comparison instead.