

Separable differential equations: growth, decay, cooling and the logistic curve

Ten separable differential equations: exponential growth with a negative start, $y' = x/y$, a solution that blows up in finite time, the equation whose solution is softplus, half-life, Newton's law of cooling, the sigmoid as the solution of $y' = y(1 - y)$, gradient flow on a quadratic loss, a constant solution lost by dividing, and a draining tank, with worked solutions and the mistakes of adding the constant too late, splitting a square root and forgetting where a solution stops.

BEFORE YOU START

A differential equation describes how a quantity changes, and solving it recovers the quantity itself. When the rate factors into a function of x times a function of y , the variables can be separated and each side integrated on its own. The sigmoid, softplus and the exponential decay of gradient flow on a quadratic loss all come out of equations like this. The work goes wrong in a few places: the constant of integration added after exponentiating instead of before, $\sqrt{x^2 + 9}$ split into $x + 3$, a constant solution thrown away by dividing by zero, the shift in Newton's law of cooling forgotten, and a formula used past the point where the solution stops. These ten problems cover each of them.

- **Separable.** $\frac{dy}{dx} = g(x) h(y)$. Divide by $h(y)$, multiply by dx , and integrate both sides: $\int \frac{dy}{h(y)} = \int g(x) dx + C$. One constant is enough, and it goes in **before** you solve for y .
- **Constant solutions.** If $h(y_0) = 0$, then $y \equiv y_0$ is a solution. Dividing by $h(y)$ assumes $h(y) \neq 0$, so check these separately.
- **Exponential growth and decay.** $y' = ky$ has solutions $y = y(0)e^{kx}$. From $\ln|y| = kx + c$ you get $y = Ae^{kx}$ with $A = \pm e^c$, and $A = 0$ is the constant solution, so A can be any real number.
- **Initial condition.** Use $y(x_0) = y_0$ to find C . Then check that the answer satisfies the equation.
- **Interval of validity.** A solution lives on the interval containing x_0 where the formula is defined and differentiable. If the formula blows up, the solution ends there; the formula's values on the far side belong to a different solution.
- **Newton's law of cooling.** $T' = -k(T - T_a)$ for room temperature T_a . The quantity that decays exponentially is $T - T_a$, not T .

Builds on: Integration: substitution and parts, Partial fractions and integrating rational functions

Problems

1. Solve $\frac{dy}{dx} = 3y$ with $y(0) = -2$.

difficulty ·

2. Solve $\frac{dy}{dx} = \frac{x}{y}$ with $y(0) = 3$.

difficulty ·

3. Solve $\frac{dy}{dx} = xy^2$ with $y(0) = 1$, and find the interval on which the solution exists.

difficulty ··

4. Solve $\frac{dy}{dx} = e^{x-y}$ with $y(0) = \ln 2$.

difficulty ··

5. A radioactive sample decays at a rate proportional to its mass, $\frac{dm}{dt} = -km$, with a half-life of 5 years. How long until 10% of it is left?

difficulty ··

6. Coffee at 90°C is left in a 20°C room, and $T' = -k(T - 20)$. After 10 minutes it is at 60°C . When does it reach 30°C ?

difficulty ··

7. Solve $\frac{dy}{dx} = y(1 - y)$ with $y(0) = \frac{1}{2}$.

difficulty ···

8. Gradient flow is gradient descent with an infinitely small step: $\frac{dw}{dt} = -\eta L'(w)$. For $L(w) = \frac{1}{2}(w - 3)^2$, $\eta = 0.1$ and $w(0) = 0$, find $w(t)$ and the time at which $w = 2.9$.

difficulty ··

9. Solve $\frac{dy}{dx} = 2x(y - 1)$, first with $y(0) = 3$, then with $y(0) = 1$.

difficulty ··

10. Water drains from a tank through a hole at the bottom, and its difficulty ...
depth h (in meters, with t in minutes) satisfies $\frac{dh}{dt} = -\frac{1}{2}\sqrt{h}$, with $h(0) = 16$. When is the
tank empty, and when is it half full?

Answers

1. $y = -2e^{3x}$
2. $y = \sqrt{x^2 + 9}$, for all x
3. $y = \frac{2}{2 - x^2}$, on $-\sqrt{2} < x < \sqrt{2}$
4. $y = \ln(1 + e^x)$, the softplus function
5. $\frac{5 \ln 10}{\ln 2} \approx 16.61$ years
6. About 34.77 minutes
7. $y = \frac{1}{1 + e^{-x}}$, the sigmoid
8. $w(t) = 3 - 3e^{-0.1t}$; $w = 2.9$ at $t = 10 \ln 30 \approx 34.01$
9. $y = 1 + 2e^{x^2}$ for $y(0) = 3$; $y \equiv 1$ for $y(0) = 1$
10. Empty at $t = 16$ minutes; half full at $t = 16 - 8\sqrt{2} \approx 4.69$ minutes

Worked solutions

Problem 1

Solve $\frac{dy}{dx} = 3y$ with $y(0) = -2$.

$\frac{dy}{y} = 3 dx$	Separate, for $y \neq 0$.
$\ln y = 3x + c$	Integrate both sides; one constant.
$ y = e^c e^{3x}$, so $y = Ae^{3x}$ with $A = \pm e^c$	The absolute value is why A can be negative.
$y(0) = A = -2$	The initial condition.
$y = -2e^{3x}$	Check: $y' = -6e^{3x} = 3y$, and $y(0) = -2$. The solution stays negative and grows in size.

Problem 2

Solve $\frac{dy}{dx} = \frac{x}{y}$ with $y(0) = 3$.

$y dy = x dx$	Multiply both sides by $y dx$.
$\frac{y^2}{2} = \frac{x^2}{2} + C$	Integrate both sides.
$C = \frac{9}{2}$, so $y^2 = x^2 + 9$	At $x = 0$, $y = 3$.
$y = \sqrt{x^2 + 9}$	Take the positive root, because $y(0) = 3 > 0$.
$y = \sqrt{x^2 + 9}$, for all x	Check: $y' = \frac{x}{\sqrt{x^2 + 9}} = \frac{x}{y}$. The curve is the upper branch of the hyperbola $y^2 - x^2 = 9$.

Problem 3

Solve $\frac{dy}{dx} = xy^2$ with $y(0) = 1$, and find the interval on which the solution exists.

$$y^{-2} dy = x dx$$

Separate, for $y \neq 0$.

$$-\frac{1}{y} = \frac{x^2}{2} + C$$

$$\int y^{-2} dy = -y^{-1}.$$

$$C = -1$$

At $x = 0$: $-1 = 0 + C$.

$$\frac{1}{y} = 1 - \frac{x^2}{2} = \frac{2 - x^2}{2}, \text{ so } y = \frac{2}{2 - x^2}$$

Solve for y only after C is in.

$$y = \frac{2}{2 - x^2}, \text{ on } -\sqrt{2} < x < \sqrt{2}$$

The formula blows up at $x = \pm\sqrt{2} \approx \pm 1.414$, so the solution through $y(0) = 1$ exists only between them. Growth at a rate y^2 is faster than exponential and reaches infinity in finite time.

Problem 4

Solve $\frac{dy}{dx} = e^{x-y}$ with $y(0) = \ln 2$.

$$\frac{dy}{dx} = e^x e^{-y}, \text{ so } e^y dy = e^x dx$$

Split the exponential, then separate.

$$e^y = e^x + C$$

Integrate both sides.

$$C = 1$$

At $x = 0$: $e^{\ln 2} = 2 = 1 + C$.

$$y = \ln(1 + e^x)$$

Take the log; $1 + e^x > 0$, so this is defined for every x .

$$y = \ln(1 + e^x), \text{ the softplus function}$$

Check: $y' = \frac{e^x}{1+e^x}$ and $e^{x-y} = \frac{e^x}{1+e^x}$. The derivative of softplus is the sigmoid.

Problem 5

A radioactive sample decays at a rate proportional to its mass, $\frac{dm}{dt} = -km$, with a half-life of 5 years. How long until 10% of it is left?

$m = m_0 e^{-kt}$	The solution of $m' = -km$, as in Problem 1.
$e^{-5k} = \frac{1}{2}$, so $k = \frac{\ln 2}{5}$	After one half-life, half the mass is left.
$e^{-kt} = \frac{1}{10}$, so $t = \frac{\ln 10}{k} = \frac{5 \ln 10}{\ln 2}$	Solve for the time when $m = 0.1m_0$.
$\frac{5 \ln 10}{\ln 2} \approx 16.61$ years	About 3.32 half-lives: $2^{-3.32} \approx 0.1$. The initial mass cancels, so it was never needed.

Problem 6

Coffee at 90°C is left in a 20°C room, and $T' = -k(T - 20)$. After 10 minutes it is at 60°C . When does it reach 30°C ?

$\frac{dT}{T - 20} = -k dt$, so $T - 20 = Ae^{-kt}$	Separate and integrate; $T - 20$ decays exponentially.
$A = 90 - 20 = 70$	At $t = 0$.
$70e^{-10k} = 40$, so $k = \frac{\ln(7/4)}{10} \approx 0.0560$ per minute	At $t = 10$, $T - 20 = 40$.
$70e^{-kt} = 10$, so $t = \frac{\ln 7}{k} = \frac{10 \ln 7}{\ln(7/4)}$	At $T = 30$, $T - 20 = 10$.
About 34.77 minutes	T approaches 20°C but never reaches it.

Problem 7

Solve $\frac{dy}{dx} = y(1 - y)$ with $y(0) = \frac{1}{2}$.

$$\frac{dy}{y(1-y)} = dx, \text{ and } \frac{1}{y(1-y)} = \frac{1}{y} + \frac{1}{1-y}$$

Separate, then partial fractions by the cover-up rule.

$$\ln|y| - \ln|1-y| = x + c \qquad \int \frac{dy}{1-y} = -\ln|1-y|.$$

$$\frac{y}{1-y} = Ae^x, \text{ and } A = 1 \qquad \text{At } x = 0, \frac{1/2}{1/2} = 1.$$

$$y = e^x(1-y), \text{ so } y(1+e^x) = e^x \text{ and } y = \frac{e^x}{1+e^x}$$

Solve for y .

$$y = \frac{1}{1+e^{-x}}, \text{ the sigmoid}$$

Divide top and bottom by e^x . This is why $\sigma' = \sigma(1 - \sigma)$, and why logistic growth with rate 1 and capacity 1 is a sigmoid. The constant solutions $y \equiv 0$ and $y \equiv 1$ are its two horizontal asymptotes.

Problem 8

Gradient flow is gradient descent with an infinitely small step: $\frac{dw}{dt} = -\eta L'(w)$. For $L(w) = \frac{1}{2}(w-3)^2$, $\eta = 0.1$ and $w(0) = 0$, find $w(t)$ and the time at which $w = 2.9$.

$$\frac{dw}{dt} = -0.1(w-3) \qquad L'(w) = w-3.$$

$$w-3 = Ae^{-0.1t}, \text{ and } A = -3 \qquad \text{The same structure as Problem 6: } w-3 \text{ decays exponentially; } w(0)-3 = -3.$$

$$w = 3 - 3e^{-0.1t} \qquad w \text{ climbs toward the minimizer 3.}$$

$$3e^{-0.1t} = 0.1, \text{ so } t = 10 \ln 30 \qquad w = 2.9 \text{ means the gap is 0.1.}$$

$$w(t) = 3 - 3e^{-0.1t}; w = 2.9 \text{ at } t = 10 \ln 30 \approx 34.01$$

On a quadratic loss the distance to the minimum shrinks by the same factor in every unit of time. Gradient descent with step η shrinks it by $1 - \eta$ per step, and $(1 - \eta)^n \approx e^{-\eta n}$.

Problem 9

Solve $\frac{dy}{dx} = 2x(y - 1)$, first with $y(0) = 3$, then with $y(0) = 1$.

$$\frac{dy}{y - 1} = 2x \, dx, \text{ for } y \neq 1$$

Separate.

$$\ln|y - 1| = x^2 + c, \text{ so } y = 1 + Ae^{x^2}$$

$A = \pm e^c$ is not 0 so far.

$$y(0) = 3 \text{ gives } A = 2$$

$$1 + A = 3.$$

For $y(0) = 1$: $y \equiv 1$ satisfies $y' = 0 = 2x(1 - 1)$

The constant solution, which step 1 excluded by dividing by $y - 1$. It is the case $A = 0$.

$$y = 1 + 2e^{x^2} \text{ for } y(0) = 3; y \equiv 1 \text{ for } y(0) = 1$$

Check: $y' = 4xe^{x^2} = 2x \cdot 2e^{x^2} = 2x(y - 1)$.

Problem 10

Water drains from a tank through a hole at the bottom, and its depth h (in meters, with t in minutes) satisfies $\frac{dh}{dt} = -\frac{1}{2}\sqrt{h}$, with $h(0) = 16$. When is the tank empty, and when is it half full?

$$h^{-1/2} \, dh = -\frac{1}{2} \, dt$$

Separate, for $h > 0$.

$$2\sqrt{h} = -\frac{t}{2} + C, \text{ and } C = 8$$

$$\int h^{-1/2} \, dh = 2h^{1/2}; \text{ at } t = 0, 2 \cdot 4 = 8.$$

$$\sqrt{h} = 4 - \frac{t}{4}, \text{ so } h = \left(4 - \frac{t}{4}\right)^2 \text{ for } 0 \leq t \leq 16$$

Valid while $\sqrt{h} \geq 0$.

$h = 0$ at $t = 16$	After that $h \equiv 0$: the formula $\left(4 - \frac{t}{4}\right)^2$ would start rising again, which an empty tank does not do.
$\sqrt{8} = 4 - \frac{t}{4}$, so $t = 16 - 4\sqrt{8} = 16 - 8\sqrt{2}$	Half full means $h = 8$, if the tank has straight sides.
Empty at $t = 16$ minutes; half full at $t = 16 - 8\sqrt{2} \approx 4.69$ minutes	The first half drains in under a third of the time, because the flow is fastest when the water is deepest.

Where this goes wrong

1. Adding the constant after exponentiating

The $+C$ is often left until the end, as it would be for an ordinary antiderivative.

$\frac{dy}{y} = 3 dx$, so $\ln y = 3x$	Right so far, apart from the constant: Problem 1, step 2.
“Solve for y , then add the constant.”	The habit that causes the mistake: the constant belongs to the integration step, not to the final line.
$y = e^{3x} + C$, and $y(0) = -2$ gives $y = e^{3x} - 3$ ✗	Check it: $y' = 3e^{3x}$, but $3y = 3e^{3x} - 9$, so it does not satisfy the equation. Added at the integration step, $\ln y = 3x + c$ becomes $y = Ae^{3x}$, a multiplicative constant, and $y = -2e^{3x}$.

2. Splitting a square root

After solving for y^2 , taking the square root looks like taking it term by term.

$y^2 = x^2 + 9$	Right so far: Problem 2, step 3.
“The square root of x^2 is x and of 9 is 3.”	The habit that causes the mistake: $\sqrt{a+b}$ treated as $\sqrt{a} + \sqrt{b}$.

$$y = x + 3 \quad \times$$

At $x = 4$ this gives 7, but $\sqrt{16 + 9} = 5$.
 Check the equation too: $y' = 1$, while
 $\frac{x}{y} = \frac{x}{x+3}$, so $y = x + 3$ is not a solution.
 The answer is $y = \sqrt{x^2 + 9}$.

3. Losing the constant solution

Separating means dividing by the function of y , and that function can be 0.

$$\frac{dy}{y-1} = 2x \, dx, \text{ so } y = 1 + Ae^{x^2} \text{ with } A = \pm e^c$$

Right so far: Problem 9, step 2.

“ e^c is never 0, so $A \neq 0$.”

The habit that causes the mistake: only the solutions the division allowed are kept.

With $y(0) = 1$, $1 + A = 1$ needs $A = 0$, so there is no solution $\times$

$y \equiv 1$ is a solution: both sides of $y' = 2x(y-1)$ are 0. Dividing by $y-1$ excluded it. Whenever you divide by $h(y)$, check the values with $h(y) = 0$ separately; here they make $A = 0$ allowed.

4. Forgetting the room temperature in Newton's law

Exponential decay is usually $y = y_0 e^{-kt}$, and that pattern gets applied to T itself.

$$T' = -k(T - 20), T(0) = 90, T(10) = 60$$

Right so far: Problem 6.

“ T decays exponentially, so $T = 90e^{-kt}$.”

The habit that causes the mistake: the shift by the room temperature is dropped.

$$90e^{-10k} = 60 \text{ gives } k = \frac{\ln 1.5}{10}, \text{ and } T = 30 \text{ at } t = \frac{10 \ln 3}{\ln 1.5} \approx 27.10 \text{ minutes } \times$$

$T = 90e^{-kt}$ decays to 0°C , not to the room's 20°C . It is $T - 20$ that decays: $70e^{-10k} = 40$, and the coffee reaches 30°C at about 34.77 minutes.

5. Using the formula past the blow-up

The formula for the solution is defined on both sides of the point where it blows up.

$$y = \frac{2}{2 - x^2} \text{ solves } y' = xy^2 \text{ with } y(0) = 1$$

Right so far: Problem 3, step 4.

“Plug in $x = 2$.”

The habit that causes the mistake: the formula treated as valid wherever it can be evaluated.

$$y(2) = \frac{2}{2 - 4} = -1 \text{ ✗}$$

The solution through $y(0) = 1$ goes to $+\infty$ at $x = \sqrt{2} \approx 1.414$ and stops there; a solution must be differentiable on an interval, and it cannot jump from $+\infty$ to negative values. The branch beyond $\sqrt{2}$ is a different solution, one that never passes through $(0, 1)$. $y(2)$ is not defined for this problem.

eulerpad.app/practice/separable-differential-equations