# Quadratic forms and positive definite matrices

Ten problems on quadratic forms and positive definite matrices: writing a form as $x^T Ax$ with a symmetric matrix, classifying definite, semidefinite and indefinite matrices by their eigenvalues, the leading principal minors of a 3×3 matrix, completing the square as $LDL^T$, a Cholesky factor by hand, the range of a parameter that keeps a matrix positive definite, the largest and smallest values on the unit circle, the axes of an ellipse, why $A^TA$ is positive semidefinite and when it is definite, and the Hessian of a ridge regression loss, with worked solutions and the mistakes of the unhalved cross term, a positive diagonal taken as proof, leading minors for a semidefinite test, the long axis along the largest eigenvalue and the extremes read off the diagonal.

**BEFORE YOU START**

A quadratic form is a sum of squares and cross products, such as $3x^2 + 4xy + 2y^2$, and every one of them can be written as $x^\top Ax$ with a symmetric matrix $A$. The question that matters is its sign: a form that is positive for every nonzero $x$ is positive definite, and that single property is what makes a critical point a minimum, a loss convex with a unique minimizer, a covariance matrix valid and a Cholesky factor exist. These ten problems build the matrix of a form, test definiteness three ways (eigenvalues, leading principal minors and completing the square), factor a matrix by Cholesky, find the extremes of a form on the unit circle and the axes of an ellipse, and end with why ridge regression always has exactly one solution.

- A **quadratic form** is $q(x) = x^\top Ax = \sum_{i,j} A_{ij} x_i x_j$ with $A$ symmetric. The coefficient of $x_i^2$ is $A_{ii}$, and the coefficient of $x_i x_j$ for $i \neq j$ is $2A_{ij}$, so the off-diagonal entries are half the cross coefficients.
- $A$ is **positive definite** (PD) if $x^\top Ax > 0$ for every $x \neq 0$, **positive semidefinite** (PSD) if $x^\top Ax \geq 0$ for every $x$, negative (semi)definite if $-A$ is, and **indefinite** if $x^\top Ax$ takes both signs.
- **Eigenvalue test.** A symmetric $A$ is $Q\Lambda Q^\top$ with $Q$ orthogonal, so with $y = Q^\top x$, $x^\top Ax = \sum_i \lambda_i y_i^2$. $A$ is PD exactly when every $\lambda_i > 0$, PSD when every $\lambda_i \geq 0$, and indefinite when there are eigenvalues of both signs.
- **Sylvester's criterion.** $A$ is PD exactly when its leading principal minors (the determinants of the top-left $1 \times 1$, $2 \times 2$, ..., $n \times n$ blocks) are all positive. For semidefiniteness the leading minors are not enough: *every* principal minor must be $\geq 0$.
- **Cholesky.** $A$ is PD exactly when $A = LL^\top$ with $L$ lower-triangular and its diagonal positive. Equivalently $A = LDL^\top$ with $L$ unit lower-triangular and the pivots of elimination on the diagonal of $D$, all positive.

- **Rayleigh quotient.** For symmetric $A$, the largest and smallest values of $x^\top Ax$ over unit vectors $x$ are $\lambda_{\max}$ and $\lambda_{\min}$, reached at their eigenvectors. For PD $A$, the set $x^\top Ax = c$ is an ellipse (ellipsoid) with axes along the eigenvectors and semi-axis lengths $\sqrt{c/\lambda_i}$.
- For any matrix $M$, $v^\top M^\top Mv = \|Mv\|^2$.

Builds on: Eigenvalues and eigenvectors by hand, Gaussian elimination, inverses and determinants by hand

## Problems

**1.** Write $q(x,y) = 3x^2 + 4xy + 2y^2$ as $x^\top Ax$ with $A$ symmetric, and check it at $(1,-1)$. Then write the matrix of $x^2 + 2y^2 + 3z^2 + 2xy - 4yz$. difficulty ·

**2.** Classify each matrix as positive definite, positive semidefinite or indefinite from its eigenvalues: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$. difficulty ·

**3.** Use the leading principal minors to show that $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ is positive definite, and confirm with its eigenvalues. difficulty ··

**4.** Complete the square in $q(x,y) = 3x^2 + 4xy + 2y^2$, and write the result as $A = LDL^\top$. difficulty ··

**5.** Find the Cholesky factor $L$ of $A = \begin{pmatrix} 4 & 2 & -2 \\ 2 & 10 & 2 \\ -2 & 2 & 6 \end{pmatrix}$, and use it to get $\det A$. difficulty ··

**6.** For which real $k$ is $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & k \\ 0 & k & 2 \end{pmatrix}$ positive definite? difficulty ··

**7.** Find the largest and smallest values of $q(x,y) = 5x^2 + 4xy + 2y^2$ on the unit circle $x^2 + y^2 = 1$, and where they occur. difficulty ··

**8.** Find the axes, the semi-axis lengths and the area of the ellipse $5x^2 - 4xy + 5y^2 = 21$. difficulty ··

**9.** Show that $M^\top M$ is positive semidefinite for any matrix $M$, and positive definite exactly when the columns of $M$ are independent. Test $M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{pmatrix}$. difficulty ···

**10.** Ridge regression minimizes $L(w) = \|Xw - y\|^2 + \lambda \|w\|^2$. Find the gradient and Hessian of $L$. With $X = M_2$ from Problem 9, $y = (1, 2, 3)$ and $\lambda = \tfrac{1}{2}$, show that the Hessian is positive definite and find the unique minimizer. difficulty ···

## Answers

1. $A = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}$; $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & -2 \\ 0 & -2 & 3 \end{pmatrix}$
2. $A$ is positive definite, $B$ is indefinite, $C$ is positive semidefinite (not definite)
3. $\Delta_1, \Delta_2, \Delta_3 = 2, 3, 4$; the eigenvalues are $2 - \sqrt{2} \approx 0.586$, $2$ and $2 + \sqrt{2} \approx 3.414$, all positive
4. $q = 3\big(x + \tfrac{2}{3}y\big)^2 + \tfrac{2}{3}y^2$, $A = LDL^\top$ with $L = \begin{pmatrix} 1 & 0 \\ \frac{2}{3} & 1 \end{pmatrix}$, $D = \operatorname{diag}(3, \tfrac{2}{3})$: positive definite
5. $L = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ -1 & 1 & 2 \end{pmatrix}$, $\det A = (2 \cdot 3 \cdot 2)^2 = 144$
6. $-\sqrt{3} < k < \sqrt{3}$
7. Maximum $6$ at $\pm\frac{1}{\sqrt{5}}(2, 1)$; minimum $1$ at $\pm\frac{1}{\sqrt{5}}(1, -2)$
8. Major semi-axis $\sqrt{7}$ along $(1, 1)$, minor semi-axis $\sqrt{3}$ along $(1, -1)$; area $\pi\sqrt{7}\sqrt{3} = \pi\sqrt{21} \approx 14.40$
9. $M^\top M$ is always PSD; $M_1^\top M_1$ is PD, $M_2^\top M_2$ is only PSD, with $v^\top M_2^\top M_2 v = 0$ at $v = (2, -1)$
10. $\nabla^2 L = 2(X^\top X + \lambda I)$, PD with eigenvalues $1$ and $31$; $w^* = \big(\frac{12}{31}, \frac{24}{31}\big)$

# Worked solutions

## Problem 1

> Write $q(x, y) = 3x^2 + 4xy + 2y^2$ as $x^\top A x$ with $A$ symmetric, and check it at $(1, -1)$. Then write the matrix of $x^2 + 2y^2 + 3z^2 + 2xy - 4yz$.

| | |
|---|---|
| $A_{11} = 3$ and $A_{22} = 2$ | The squares go on the diagonal. |
| $2A_{12} = 4$, so $A_{12} = A_{21} = 2$ | The cross term $4xy$ is $A_{12}xy + A_{21}yx$, split evenly between the two off-diagonal entries. |
| $A = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}$ | Symmetric, as a form's matrix should be. |
| At $(1, -1)$: $q = 3 - 4 + 2 = 1$, and $x^\top A x = (1, -1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$ | $Ax = (3 - 2, 2 - 2) = (1, 0)$. Using $4$ off the diagonal instead gives $-3$ (Mistake 1). |
| $A = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}$; $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & -2 \\ 0 & -2 & 3 \end{pmatrix}$ | In the second form, $2xy$ gives $A_{12} = 1$, $-4yz$ gives $A_{23} = -2$, and there is no $xz$ term, so $A_{13} = 0$. |

## Problem 2

> Classify each matrix as positive definite, positive semidefinite or indefinite from its eigenvalues: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$.

| | |
|---|---|
| $A$: $\lambda^2 - 4\lambda + 3 = 0$, so $\lambda = 1, 3$ | Trace $4$, determinant $3$. |
| $B$: $\lambda^2 - 2\lambda - 3 = 0$, so $\lambda = 3, -1$ | Trace $2$, determinant $-3$. A negative determinant of a $2 \times 2$ always means one eigenvalue of each sign. |
| $C$: $\lambda^2 - 5\lambda = 0$, so $\lambda = 0, 5$ | Determinant $0$: the rows are proportional. |

| | |
|---|---|
| $q_B(1,-1) = 1 - 4 + 1 = -2$ and $q_C(2,-1) = 4 - 8 + 4 = 0$ | A witness for each: $(1,-1)$ is $B$'s eigenvector for $-1$, and $(2,-1)$ spans $C$'s null space. |
| **$A$ is positive definite, $B$ is indefinite, $C$ is positive semidefinite (not definite)** | $B$ has positive diagonal entries and is still not PD (Mistake 2). |

### Problem 3

Use the leading principal minors to show that $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ is positive definite, and confirm with its eigenvalues.

| | |
|---|---|
| $\Delta_1 = 2$ | The top-left entry. |
| $\Delta_2 = 2 \cdot 2 - (-1)(-1) = 3$ | The top-left $2 \times 2$ block. |
| $\Delta_3 = 2(4-1) - (-1)(-2-0) + 0 = 6 - 2 = 4$ | Expand along the first row. |
| All three are positive, so $A$ is PD by Sylvester's criterion | This is the second-difference matrix behind a discretized $-u''$. |
| $\Delta_1, \Delta_2, \Delta_3 = 2, 3, 4$; **the eigenvalues are** $2 - \sqrt{2} \approx 0.586$, $2$ **and** $2 + \sqrt{2} \approx 3.414$, **all positive** | Their product is $4 = \det A$ and their sum is $6 = \operatorname{tr} A$. |

### Problem 4

Complete the square in $q(x,y) = 3x^2 + 4xy + 2y^2$, and write the result as $A = LDL^\top$.

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| $3x^2 + 4xy = 3\big(x^2 + \tfrac{4}{3}xy\big) = 3\big(x + \tfrac{2}{3}y\big)^2 - \tfrac{4}{3}y^2$ | Factor out the $x^2$ coefficient, then half of $\tfrac{4}{3}$ is $\tfrac{2}{3}$, and $3 \cdot \tfrac{4}{9} = \tfrac{4}{3}$. |

| | |
|---|---|
| $q = 3\left(x + \frac{2}{3}y\right)^2 + \frac{2}{3}y^2$ | $2 - \frac{4}{3} = \frac{2}{3}$. |
| Both coefficients are positive, so $q > 0$ unless $y = 0$ and $x + \frac{2}{3}y = 0$, that is unless $x = y = 0$ | A sum of positive multiples of squares of independent combinations. |
| $L = \begin{pmatrix} 1 & 0 \\ \frac{2}{3} & 1 \end{pmatrix}$, $D = \operatorname{diag}\left(3, \frac{2}{3}\right)$ | The rows of $L^\top$ are the combinations being squared; the entries of $D$ are the pivots of elimination on $A$. |
| $q = 3\left(x + \frac{2}{3}y\right)^2 + \frac{2}{3}y^2$, $A = LDL^\top$ **with** $L = \begin{pmatrix} 1 & 0 \\ \frac{2}{3} & 1 \end{pmatrix}$, $D = \operatorname{diag}(3, \frac{2}{3})$: **positive definite** | Check: $\det A = 3 \cdot \frac{2}{3} = 2 = 6 - 4$. |

## Problem 5

Find the Cholesky factor $L$ of $A = \begin{pmatrix} 4 & 2 & -2 \\ 2 & 10 & 2 \\ -2 & 2 & 6 \end{pmatrix}$, and use it to get $\det A$.

| | |
|---|---|
| $l_{11} = \sqrt{4} = 2$, $l_{21} = \frac{2}{2} = 1$, $l_{31} = \frac{-2}{2} = -1$ | The first column of $A$ is $l_{11}$ times the first column of $L$. |
| $l_{22} = \sqrt{10 - l_{21}^2} = \sqrt{9} = 3$ | Subtract what the first column already supplies, then take the root. |
| $l_{32} = \dfrac{2 - l_{31}l_{21}}{l_{22}} = \dfrac{2 - (-1)(1)}{3} = 1$ | From $A_{32} = l_{31}l_{21} + l_{32}l_{22}$. |
| $l_{33} = \sqrt{6 - l_{31}^2 - l_{32}^2} = \sqrt{6 - 1 - 1} = 2$ | Every square root was of a positive number, which is the proof that $A$ is PD. |
| $L = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ -1 & 1 & 2 \end{pmatrix}$, $\det A = (2 \cdot 3 \cdot 2)^2 = 144$ | $\det A = \det L \, \det L^\top$. Sampling $\mathcal{N}(0, A)$ is $L\varepsilon$ with $\varepsilon \sim \mathcal{N}(0, I)$. |

## Problem 6

> For which real $k$ is $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & k \\ 0 & k & 2 \end{pmatrix}$ positive definite?

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|---|---|
| $\Delta_1 = 2 > 0$ and $\Delta_2 = 4 - 1 = 3 > 0$ | These do not involve $k$. |
| $\Delta_3 = 2(4 - k^2) - 1(2 - 0) + 0 = 6 - 2k^2$ | Expand along the first row. |
| $6 - 2k^2 > 0 \iff k^2 < 3$ | Sylvester's criterion needs all three minors positive. |
| At $k = \sqrt{3}$ the determinant is $0$ and the smallest eigenvalue is $0$: PSD but not PD | The boundary of the range. |
| $-\sqrt{3} < k < \sqrt{3}$ | Check $k = 2$: $(1, -2, 2)$ gives $x^\top A x = 2 - 4 + 8 + 8 - 16 = -2 < 0$. |

## Problem 7

> Find the largest and smallest values of $q(x, y) = 5x^2 + 4xy + 2y^2$ on the unit circle $x^2 + y^2 = 1$, and where they occur.

| | |
|---|---|
| $A = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$ | Half of $4$ off the diagonal. |
| $\lambda^2 - 7\lambda + 6 = 0$, so $\lambda = 6, 1$ | Trace $7$, determinant $10 - 4 = 6$. |
| $\lambda = 6$: $(A - 6I)v = 0$ with rows $(-1, 2)$, so $v = \frac{1}{\sqrt{5}}(2, 1)$ | Check: $q = \frac{1}{5}(20 + 8 + 2) = 6$. |
| $\lambda = 1$: rows $(4, 2)$, so $v = \frac{1}{\sqrt{5}}(1, -2)$ | Check: $q = \frac{1}{5}(5 - 8 + 8) = 1$. |
| **Maximum** $6$ **at** $\pm\frac{1}{\sqrt{5}}(2, 1)$; **minimum** $1$ **at** $\pm\frac{1}{\sqrt{5}}(1, -2)$ | Every other unit vector gives a value in between, for example $q(1, 0) = 5$. The smallest diagonal entry, $2$, is not the minimum (Mistake 5). |

## Problem 8

Find the axes, the semi-axis lengths and the area of the ellipse $5x^2 - 4xy + 5y^2 = 21$.

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|---|---|
| $A = \begin{pmatrix} 5 & -2 \\ -2 & 5 \end{pmatrix}$ has $\lambda = 3$ for $(1,1)/\sqrt{2}$ and $\lambda = 7$ for $(1,-1)/\sqrt{2}$ | $A(1,1) = (3,3)$ and $A(1,-1) = (7,-7)$. |
| In the rotated coordinates $y = Q^\top x$ the ellipse is $3y_1^2 + 7y_2^2 = 21$ | The cross term disappears along the eigenvectors. |
| Semi-axes $\sqrt{21/3} = \sqrt{7} \approx 2.646$ along $(1,1)$ and $\sqrt{21/7} = \sqrt{3} \approx 1.732$ along $(1,-1)$ | Divide by the eigenvalue: the *smaller* eigenvalue gives the *longer* axis. |
| Check the end of the long axis, $\sqrt{7} \cdot \frac{1}{\sqrt{2}}(1,1)$, where $x = y = \sqrt{3.5}$: $(5 - 4 + 5) \cdot 3.5 = 21$ | It lies on the ellipse. |
| **Major semi-axis $\sqrt{7}$ along** $(1,1)$, **minor semi-axis $\sqrt{3}$ along** $(1,-1)$; **area** $\pi\sqrt{7}\sqrt{3} = \pi\sqrt{21} \approx 14.40$ | Putting the long axis along the eigenvector of 7 gives a point where the form is 49 (Mistake 4). |

## Problem 9

Show that $M^\top M$ is positive semidefinite for any matrix $M$, and positive definite exactly when the columns of $M$ are independent. Test $M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{pmatrix}$.

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|---|---|
| $v^\top M^\top M v = (Mv)^\top (Mv) = \|Mv\|^2 \geq 0$ | A squared length is never negative, so $M^\top M$ is PSD. |
| $\|Mv\|^2 = 0 \iff Mv = 0$, and $Mv$ is the combination of the columns with coefficients $v$ | So $v^\top M^\top M v > 0$ for every $v \neq 0$ exactly when no nonzero combination of the columns is 0. |

| | |
|---|---|
| $M_1^\top M_1 = \begin{pmatrix} 3 & 6 \\ 6 & 14 \end{pmatrix}$, with minors 3 and $42 - 36 = 6$ | Both positive: PD, as the columns $(1, 1, 1)$ and $(1, 2, 3)$ are independent. |
| $M_2^\top M_2 = \begin{pmatrix} 3 & 6 \\ 6 & 12 \end{pmatrix}$, with determinant 0 and eigenvalues 0 and 15 | The second column is twice the first, and $v = (2, -1)$ gives $M_2 v = 0$. |
| **$M^\top M$ is always PSD; $M_1^\top M_1$ is PD, $M_2^\top M_2$ is only PSD, with $v^\top M_2^\top M_2 v = 0$ at $v = (2, -1)$** | This is why the normal equations have a unique solution exactly when the features are not collinear. |

### Problem 10

> Ridge regression minimizes $L(w) = \|Xw - y\|^2 + \lambda \|w\|^2$. Find the gradient and Hessian of $L$. With $X = M_2$ from Problem 9, $y = (1, 2, 3)$ and $\lambda = \frac{1}{2}$, show that the Hessian is positive definite and find the unique minimizer.

| | |
|---|---|
| $\nabla L = 2X^\top (Xw - y) + 2\lambda w$ and $\nabla^2 L = 2(X^\top X + \lambda I)$ | $L = w^\top (X^\top X + \lambda I) w - 2 y^\top X w + y^\top y$, a quadratic form plus a linear term. |
| $v^\top (X^\top X + \lambda I) v = \|Xv\|^2 + \lambda \|v\|^2 > 0$ for $v \neq 0$ | Whatever $X$ is, when $\lambda > 0$. So $L$ is strictly convex with one minimizer. |
| $X^\top X + \frac{1}{2} I = \begin{pmatrix} 3.5 & 6 \\ 6 & 12.5 \end{pmatrix}$, with eigenvalues 0.5 and 15.5, and $X^\top y = (6, 12)$ | The eigenvalues of $X^\top X$, 0 and 15, each move up by $\lambda$. |
| $\det = 43.75 - 36 = 7.75$, so $w_1 = \dfrac{12.5 \cdot 6 - 6 \cdot 12}{7.75} = \dfrac{12}{31}$ and $w_2 = \dfrac{3.5 \cdot 12 - 6 \cdot 6}{7.75} = \dfrac{24}{31}$ | Cramer's rule on $(X^\top X + \lambda I) w = X^\top y$. |

| | |
|---|---|
| $\nabla^2 L = 2(X^\top X + \lambda I)$, **PD with eigenvalues** $1$ **and** $31$; $w^* = \left(\frac{12}{31}, \frac{24}{31}\right)$ | Without $\lambda$ the Hessian is singular and every $w$ with $w_1 + 2w_2 = 2$ is a minimizer. Ridge picks one along $(1, 2)$, with $w_1 + 2w_2 = \frac{60}{31} \approx 1.935$, shrunk slightly below 2. |

## Where this goes wrong

### 1. The cross coefficient left unhalved

The coefficient of $xy$ is copied straight into the matrix.

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|---|---|
| $q = 3x^2 + 4xy + 2y^2$, with 3 and 2 on the diagonal | Right so far: Problem 1. |
| "The $xy$ coefficient goes off the diagonal: $A = \begin{pmatrix} 3 & 4 \\ 4 & 2 \end{pmatrix}$." | Both off-diagonal entries multiply $xy$, so together they give $8xy$. |
| The form is indefinite, since $\det A = 6 - 16 = -10 < 0$ ✗ | That matrix is the form $3x^2 + 8xy + 2y^2$. At $(1, -1)$ it gives $-3$, while $q(1, -1) = 1$. With $A_{12} = 2$ the determinant is $2 > 0$ and $q$ is positive definite (Problem 4). |

### 2. A positive diagonal taken as proof

A diagonal matrix with positive entries is positive definite, and the shortcut gets applied to every matrix.

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|---|---|
| $B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ has $B_{11} = B_{22} = 1 > 0$ | Right so far: Problem 2. A positive diagonal is *necessary*, since $e_i^\top B e_i = B_{ii}$. |
| "Every diagonal entry is positive, so every $x^\top B x$ is positive." | The cross term $4xy$ can outweigh the squares. |
| $B$ is positive definite ✗ | $q_B(1, -1) = 1 - 4 + 1 = -2$, and the eigenvalues are 3 and $-1$. A positive diagonal is not sufficient; a $2 \times 2$ also needs $\det > 0$, and here $\det B = -3$. |

## 3. Leading minors used to test semidefiniteness

Sylvester's criterion is remembered as “check the leading minors”, with $> 0$ relaxed to $\geq 0$.

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|---|---|
| $A = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$ has $\Delta_1 = 0$ and $\Delta_2 = 0$ | Right so far: both leading minors are $\geq 0$. |
| “All leading minors are non-negative, so $A$ is PSD.” | For semidefiniteness every principal minor must be checked, including the bottom-right entry $-1$. |
| $A$ is positive semidefinite ✗ | $(0,1)^\top A(0,1) = -1 < 0$: $A$ is negative semidefinite. The $1 \times 1$ principal minor $A_{22} = -1$ is the one the leading minors skip. |

## 4. The long axis along the largest eigenvalue

The biggest eigenvalue is thought of as the biggest direction, as it is for a covariance matrix.

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|---|---|
| $5x^2 - 4xy + 5y^2 = 21$ has eigenvalues 3 along $(1,1)$ and 7 along $(1,-1)$ | Right so far: Problem 8. |
| “The major semi-axis $\sqrt{7}$ points along the eigenvector of 7.” | Along an eigenvector the form is $\lambda r^2$, so a large $\lambda$ reaches 21 at a small $r$. |
| The ellipse reaches $\sqrt{7} \cdot \frac{1}{\sqrt{2}}(1,-1)$ ✗ | There $x = -y = \sqrt{3.5}$ and the form is $(5+4+5) \cdot 3.5 = 49$, not 21. The semi-axis along an eigenvector is $\sqrt{c/\lambda}$: $\sqrt{21/7} = \sqrt{3}$ along $(1,-1)$ and $\sqrt{7}$ along $(1,1)$. For a Gaussian the ellipse is $x^\top \Sigma^{-1} x = c$, whose eigenvalues are $1/\lambda$, which is why the covariance's largest eigenvalue does give the long axis. |

## 5. The extremes read off the diagonal

On the unit circle the values at $(1,0)$ and $(0,1)$ are the diagonal entries, and these are taken as the extremes.

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| $q = 5x^2 + 4xy + 2y^2$ has $q(1,0) = 5$ and $q(0,1) = 2$ | Right so far: Problem 7. |

| | |
|---|---|
| “The coordinate axes give the largest and smallest values.” | Only when the cross term is $0$, so that the axes are the eigenvectors. |
| On the unit circle, $2 \le q \le 5$ ✗ | The eigenvalues are $6$ and $1$, at $\frac{1}{\sqrt{5}}(2,1)$ and $\frac{1}{\sqrt{5}}(1,-2)$. The diagonal entries always lie between $\lambda_{\min}$ and $\lambda_{\max}$, so they never show the extremes unless the matrix is diagonal. |

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