

Partial fractions and integrating rational functions

Ten rational functions split into partial fractions and integrated: distinct linear factors by the cover-up rule, three factors, a repeated factor, an irreducible quadratic, an improper fraction that needs long division first, and the definite integral of $1/(x^2 - 1)$, with worked solutions and the mistakes that choose the wrong form or skip the division.

BEFORE YOU START

Substitution and parts do not integrate $\frac{1}{x^2-1}$, but splitting it into $\frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$ turns it into two logs. Partial fractions is that split, done systematically: the form of the answer is fixed by the factors of the denominator, and only the constants are unknown. The work goes wrong in the set-up more than in the arithmetic: a repeated or quadratic factor given too few terms, an improper fraction decomposed without dividing first, a cover-up evaluated at the wrong root. These ten problems decompose first, then integrate.

- A rational function is $\frac{P(x)}{Q(x)}$ with P and Q polynomials. It is **proper** when $\deg P < \deg Q$. If it is not, divide first: $\frac{P}{Q} = S + \frac{R}{Q}$ with $\deg R < \deg Q$, and decompose only $\frac{R}{Q}$.
- Factor Q completely into linear factors and irreducible quadratics (quadratics with no real roots, such as $x^2 + 1$). Each factor contributes terms to the form:

Factor of Q	Terms in the decomposition
$x - a$, once	$\frac{A}{x-a}$
$(x - a)^2$	$\frac{A}{x-a} + \frac{B}{(x-a)^2}$
$x^2 + px + q$, irreducible	$\frac{Bx+C}{x^2+px+q}$

- To find the constants, multiply both sides by Q and either substitute convenient values of x (the roots of Q first) or match coefficients of each power of x . The **cover-up rule** is the substitution done in one step: for a factor $x - a$ that appears once, A is $\frac{P}{Q}$ with the factor $x - a$ covered up, evaluated at $x = a$. For the factor $x + 2$, $a = -2$.
- Integrals used: $\int \frac{dx}{x-a} = \ln|x - a| + C$; $\int \frac{dx}{(x-a)^2} = -\frac{1}{x-a} + C$; $\int \frac{2x}{x^2+1} dx = \ln(x^2 + 1) + C$; $\int \frac{dx}{x^2+1} = \arctan x + C$.

Builds on: Integration: substitution and parts

Problems

1. Decompose $\frac{5x-1}{(x+1)(x-2)}$ into partial fractions.

difficulty ·

2. Decompose $\frac{x+7}{x^2-x-6}$.

difficulty ·

3. Decompose $\frac{2x^2+6x-2}{x(x-1)(x+2)}$.

difficulty ··

4. Decompose $\frac{x^2+3}{x(x-1)^2}$.

difficulty ··

5. Decompose $\frac{2x^2+3x+3}{(x+1)(x^2+1)}$.

difficulty ··

6. Write $\frac{x^3+x^2+x+3}{x^2-1}$ as a polynomial plus partial fractions.

difficulty ··

7. Find $\int \frac{5x-1}{(x+1)(x-2)} dx$.

difficulty ··

8. Evaluate $\int_2^3 \frac{dx}{x^2-1}$ exactly.

difficulty ···

9. Find $\int \frac{x^2+3}{x(x-1)^2} dx$.

difficulty ···

10. Find $\int \frac{2x^2+3x+3}{(x+1)(x^2+1)} dx$.

difficulty ···

Answers

1. $\frac{2}{x+1} + \frac{3}{x-2}$

2. $\frac{2}{x-3} - \frac{1}{x+2}$

3. $\frac{1}{x} + \frac{2}{x-1} - \frac{1}{x+2}$

4. $\frac{3}{x} - \frac{2}{x-1} + \frac{4}{(x-1)^2}$

5. $\frac{1}{x+1} + \frac{x+2}{x^2+1}$

6. $x+1 + \frac{3}{x-1} - \frac{1}{x+1}$

7. $2 \ln|x+1| + 3 \ln|x-2| + C$

8. $\frac{1}{2} \ln \frac{3}{2} \approx 0.2027$

9. $3 \ln|x| - 2 \ln|x-1| - \frac{4}{x-1} + C$

10. $\ln|x+1| + \frac{1}{2} \ln(x^2+1) + 2 \arctan x + C$

Worked solutions

Problem 1

Decompose $\frac{5x-1}{(x+1)(x-2)}$ into partial fractions.

$\frac{5x-1}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$	Two distinct linear factors, one constant over each. The fraction is proper: degree 1 over degree 2.
$5x-1 = A(x-2) + B(x+1)$	Multiply both sides by $(x+1)(x-2)$. This holds for every x .
$x=2: 9 = 3B, \text{ so } B = 3.$	At $x=2$ the A term is 0. This is the cover-up rule: cover $x-2$ and evaluate $\frac{5x-1}{x+1}$ at 2.
$x=-1: -6 = -3A, \text{ so } A = 2.$	At $x=-1$ the B term is 0.
$\frac{2}{x+1} + \frac{3}{x-2}$	Check: $2(x-2) + 3(x+1) = 5x-1$.

Problem 2

Decompose $\frac{x+7}{x^2-x-6}$.

$x^2-x-6 = (x-3)(x+2)$	Factor the denominator first: -3 and 2 multiply to -6 and add to -1 .
$\frac{x+7}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$	Two distinct linear factors.
$A = \frac{3+7}{3+2} = 2$	Cover-up for $x-3$: cover it and evaluate $\frac{x+7}{x+2}$ at its root, $x=3$.
$B = \frac{-2+7}{-2-3} = \frac{5}{-5} = -1$	Cover-up for $x+2$: its root is $x=-2$, so evaluate $\frac{x+7}{x-3}$ there.
$\frac{2}{x-3} - \frac{1}{x+2}$	Check: $2(x+2) - (x-3) = x+7$.

Problem 3

Decompose $\frac{2x^2 + 6x - 2}{x(x-1)(x+2)}$.

$$\frac{2x^2 + 6x - 2}{x(x-1)(x+2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}$$

Three distinct linear factors; degree 2 over degree 3, so proper.

$$A = \frac{-2}{(0-1)(0+2)} = \frac{-2}{-2} = 1$$

Cover up x and evaluate the rest at $x = 0$.

$$B = \frac{2+6-2}{1 \cdot (1+2)} = \frac{6}{3} = 2$$

Cover up $x-1$ and evaluate at $x = 1$.

$$C = \frac{8-12-2}{(-2)(-2-1)} = \frac{-6}{6} = -1$$

Cover up $x+2$ and evaluate at $x = -2$.

$$\frac{1}{x} + \frac{2}{x-1} - \frac{1}{x+2}$$

Check: $(x-1)(x+2) + 2x(x+2) - x(x-1) = (x^2 + x - 2) + (2x^2 + 4x) - (x^2 - x) = 2x^2 + 6x - 2$.

Problem 4

Decompose $\frac{x^2 + 3}{x(x-1)^2}$.

$$\frac{x^2 + 3}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

The repeated factor $(x-1)^2$ needs a term for each power, $x-1$ and $(x-1)^2$. Three unknowns for a cubic denominator.

$$x^2 + 3 = A(x-1)^2 + Bx(x-1) + Cx$$

Multiply both sides by $x(x-1)^2$.

$$x = 0: 3 = A. \quad x = 1: 4 = C.$$

Each root kills every term but one.

$$\text{Coefficient of } x^2: 1 = A + B, \text{ so } B = -2.$$

No root isolates B , so match the x^2 terms: $A(x-1)^2$ gives Ax^2 , $Bx(x-1)$ gives Bx^2 , and Cx has none.

$$\frac{3}{x} - \frac{2}{x-1} + \frac{4}{(x-1)^2}$$

Check at $x = 2$: the left side is $\frac{7}{2}$ and the right side is $\frac{3}{2} - 2 + 4 = \frac{7}{2}$.

Problem 5

Decompose $\frac{2x^2 + 3x + 3}{(x + 1)(x^2 + 1)}$.

$$\frac{2x^2 + 3x + 3}{(x + 1)(x^2 + 1)} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 1}$$

$x^2 + 1$ has no real roots, so it stays as a factor, and its numerator is a general linear polynomial $Bx + C$, not a constant.

$$2x^2 + 3x + 3 = A(x^2 + 1) + (Bx + C)(x + 1)$$

Multiply both sides by $(x + 1)(x^2 + 1)$.

$$x = -1: 2 - 3 + 3 = 2A, \text{ so } A = 1.$$

The root of the linear factor.

$$x^2: 2 = A + B, \text{ so } B = 1. \text{ Constant: } 3 = A + C, \text{ so } C = 2.$$

Match coefficients. $(Bx + C)(x + 1) = Bx^2 + (B + C)x + C$.

$$\frac{1}{x + 1} + \frac{x + 2}{x^2 + 1}$$

Check the x coefficient, which was not used: $B + C = 3$, as required.

Problem 6

Write $\frac{x^3 + x^2 + x + 3}{x^2 - 1}$ as a polynomial plus partial fractions.

The fraction is improper: degree 3 over degree 2.

Divide before decomposing.

$$x^3 + x^2 + x + 3 = (x + 1)(x^2 - 1) + (2x + 4)$$

Long division: $x \cdot (x^2 - 1) = x^3 - x$ leaves $x^2 + 2x + 3$; $1 \cdot (x^2 - 1)$ leaves $2x + 4$, of degree less than 2. Quotient $x + 1$, remainder $2x + 4$.

$$\frac{x^3 + x^2 + x + 3}{x^2 - 1} = x + 1 + \frac{2x + 4}{(x - 1)(x + 1)}$$

Step 2 divided by $x^2 - 1$.

$$\frac{2x + 4}{(x - 1)(x + 1)} = \frac{3}{x - 1} - \frac{1}{x + 1}$$

Cover-up: at $x = 1$, $\frac{6}{2} = 3$; at $x = -1$, $\frac{2}{-2} = -1$.

$$x + 1 + \frac{3}{x - 1} - \frac{1}{x + 1}$$

Check at $x = 2$: the left side is $\frac{17}{3}$ and the right side is $3 + 3 - \frac{1}{3} = \frac{17}{3}$.

Problem 7

Find $\int \frac{5x - 1}{(x + 1)(x - 2)} dx$.

$$\int \left(\frac{2}{x + 1} + \frac{3}{x - 2} \right) dx$$

The decomposition from Problem 1.

$$\int \frac{2}{x + 1} dx = 2 \ln |x + 1| \text{ and } \int \frac{3}{x - 2} dx = 3 \ln |x - 2|.$$

$\int \frac{dx}{x - a} = \ln |x - a|$, with the constant carried along. The absolute value keeps the log defined on both sides of the root.

$$2 \ln |x + 1| + 3 \ln |x - 2| + C$$

Check by differentiating: $\frac{2}{x + 1} + \frac{3}{x - 2}$, which is the integrand.

Problem 8

Evaluate $\int_2^3 \frac{dx}{x^2 - 1}$ exactly.

$$\frac{1}{(x - 1)(x + 1)} = \frac{1/2}{x - 1} - \frac{1/2}{x + 1}$$

Cover-up: at $x = 1, \frac{1}{2}$; at $x = -1, \frac{1}{-2}$.

$$\int \frac{dx}{x^2 - 1} = \frac{1}{2} (\ln |x - 1| - \ln |x + 1|) = \frac{1}{2} \ln \left| \frac{x - 1}{x + 1} \right|$$

Integrate each term, then combine with $\ln p - \ln q = \ln \frac{p}{q}$.

$$\text{At } x = 3: \frac{1}{2} \ln \frac{2}{4} = \frac{1}{2} \ln \frac{1}{2}. \text{ At } x = 2: \frac{1}{2} \ln \frac{1}{3}.$$

Neither root of $x^2 - 1$ lies in $[2, 3]$, so the integrand is continuous there and the antiderivative can be evaluated at the ends.

$$\frac{1}{2} \ln \frac{1}{2} - \frac{1}{2} \ln \frac{1}{3} = \frac{1}{2} \ln \frac{1/2}{1/3}$$

Upper value minus lower value, then the same log rule.

$$\frac{1}{2} \ln \frac{3}{2} \approx 0.2027$$

Sanity check: the integrand falls from $\frac{1}{3}$ to $\frac{1}{8}$ on an interval of length 1, so the integral lies between 0.125 and 0.333.

Problem 9

Find $\int \frac{x^2 + 3}{x(x-1)^2} dx$.

$$\int \left(\frac{3}{x} - \frac{2}{x-1} + \frac{4}{(x-1)^2} \right) dx$$

The decomposition from Problem 4.

$$\int \frac{3}{x} dx = 3 \ln |x| \text{ and } \int \frac{-2}{x-1} dx = -2 \ln |x-1|.$$

The two linear terms give logs.

$$\int \frac{4}{(x-1)^2} dx = 4 \int (x-1)^{-2} dx = -\frac{4}{x-1}$$

The squared term is a power, not a log:
 $\int u^{-2} du = -u^{-1}$ with $u = x-1$.

$$3 \ln |x| - 2 \ln |x-1| - \frac{4}{x-1} + C$$

Check the last term: $\frac{d}{dx}(-4(x-1)^{-1}) = 4(x-1)^{-2}$.

Problem 10

Find $\int \frac{2x^2 + 3x + 3}{(x+1)(x^2+1)} dx$.

$$\int \left(\frac{1}{x+1} + \frac{x}{x^2+1} + \frac{2}{x^2+1} \right) dx$$

The decomposition from Problem 5, with $\frac{x+2}{x^2+1}$ split into its x part and its constant part, which integrate differently.

$$\int \frac{x}{x^2+1} dx = \frac{1}{2} \ln(x^2+1)$$

Substitution $u = x^2 + 1$, $du = 2x dx$: the numerator is half of du . No absolute value is needed because $x^2 + 1 > 0$.

$$\int \frac{2}{x^2+1} dx = 2 \arctan x$$

The constant part gives the arctangent.

$$\ln|x+1| + \frac{1}{2} \ln(x^2+1) + 2 \arctan x + C \quad \text{Steps 2 and 3 plus } \int \frac{dx}{x+1} = \ln|x+1|.$$

Where this goes wrong

1. Decomposing an improper fraction without dividing

The cover-up rule produces numbers for any fraction, proper or not, so nothing seems to go wrong.

$$\frac{x^3 + x^2 + x + 3}{(x-1)(x+1)}$$

Right so far: Problem 6 with the denominator factored.

“Two linear factors, so $\frac{A}{x-1} + \frac{B}{x+1}$; cover up to get $A = \frac{6}{2} = 3$ and $B = \frac{2}{-2} = -1$.”

The habit that causes the mistake: going straight to the form without comparing degrees.

$$\frac{x^3 + x^2 + x + 3}{x^2 - 1} = \frac{3}{x-1} - \frac{1}{x+1} \quad \times$$

The right side goes to 0 as $x \rightarrow \infty$ while the left side grows like x , so they cannot be equal; at $x = 2$ they are $\frac{17}{3}$ and $\frac{8}{3}$. A proper-fraction form can only match a proper fraction. Divide first; the missing piece is the quotient $x + 1$ (Problem 6).

2. One term for a repeated factor

$(x-1)^2$ looks like a single factor, so it gets a single fraction.

$$\frac{x^2 + 3}{x(x-1)^2}$$

Right so far: Problem 4.

“One fraction per factor: x and $(x-1)^2$.”

The habit that causes the mistake: counting factors, not powers.

$$\frac{x^2 + 3}{x(x-1)^2} = \frac{A}{x} + \frac{B}{(x-1)^2} \quad \times$$

Multiplying out gives $x^2 + 3 = A(x-1)^2 + Bx$. The roots give $A = 3$ and $B = 4$, but then the x^2 coefficients say $1 = A = 3$: no constants work. A repeated factor needs one term for each power, $\frac{B}{x-1} + \frac{C}{(x-1)^2}$ (Problem 4).

3. A constant over an irreducible quadratic

Every linear factor gets a constant on top, and that pattern gets carried to $x^2 + 1$.

$$\frac{2x^2 + 3x + 3}{(x + 1)(x^2 + 1)}$$

Right so far: Problem 5.

“Put a constant over each factor.”

The habit that causes the mistake: the numerator over a quadratic needs one degree less than the quadratic, which is degree 1, not 0.

$$\frac{2x^2 + 3x + 3}{(x + 1)(x^2 + 1)} = \frac{A}{x + 1} + \frac{C}{x^2 + 1} \quad \times$$

Matching coefficients of $2x^2 + 3x + 3 = A(x^2 + 1) + C(x + 1)$ gives $A = 2$ from x^2 , $C = 3$ from x , and then $3 = A + C = 5$ from the constants: no solution. With $\frac{Bx+C}{x^2+1}$ there are three unknowns for three equations (Problem 5).

4. Evaluating the cover-up at the wrong root

The factor is $x + 2$, and 2 is the number in it.

$$\frac{x + 7}{(x - 3)(x + 2)} = \frac{A}{x - 3} + \frac{B}{x + 2}, \text{ with } A = 2$$

Right so far: Problem 2, steps 2 and 3.

“Cover up $x + 2$ and put in $x = 2$.”

The habit that causes the mistake: reading the number off the factor without its sign.

$$B = \frac{2 + 7}{2 - 3} = -9 \quad \times$$

The cover-up works because the covered factor is 0 at the chosen x , and $x + 2$ is 0 at $x = -2$, not 2. At $x = -2$: $B = \frac{5}{-5} = -1$. A quick check catches it: $2(x + 2) - 9(x - 3) = -7x + 31$, not $x + 7$.

5. Integrating $1/(x - 1)^2$ as a log

Every other term in the decomposition integrates to a log, so this one does too.

$$\int \left(\frac{3}{x} - \frac{2}{x - 1} + \frac{4}{(x - 1)^2} \right) dx$$

Right so far: Problem 9, step 1.

“ $\int \frac{du}{u} = \ln|u|$ with $u = (x - 1)^2$.”

The habit that causes the mistake: $\int \frac{du}{u}$ needs du on top, and $du = 2(x - 1) dx$ is not there.

$$\int \frac{4}{(x - 1)^2} dx = 4 \ln(x - 1)^2 \quad \times$$

Differentiating the claimed answer gives $\frac{8}{x-1}$, not $\frac{4}{(x-1)^2}$. The term is the power $4(x - 1)^{-2}$, whose integral is $-\frac{4}{x-1}$ (Problem 9).

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