

Parametric and polar curves: slopes, arc length and area

Ten problems on curves given by a parameter or in polar coordinates: the slope and tangent lines of $x = t^2$, $y = t^3 - 3t$, its second derivative, an ellipse and its area, an arc length with a perfect square under the root, the cycloid's length and area, a path that runs along a segment and back, the circle $r = 2 \cos \theta$, the cardioid's area and length, the region between two polar curves, and the slope of a polar curve, with worked solutions and the mistakes of stopping at d/dt of dy/dx , dropping the absolute value in a speed and integrating over a curve traced twice.

BEFORE YOU START

Some curves are easier to describe by where a point is at each moment, $x(t)$ and $y(t)$, than by an equation in x and y : a wheel's rim, a projectile, a planet. Others are easier to describe by distance and direction from a center, $r(\theta)$. The calculus is the same as for $y = f(x)$, with the chain rule doing the translation: slopes are ratios of rates, lengths add up $\sqrt{dx^2 + dy^2}$, and areas are sums of thin strips or thin wedges. The work goes wrong in a few places: $\frac{d}{dt} \left(\frac{dy}{dx} \right)$ taken as the second derivative, an absolute value dropped from a speed, a curve traced twice counted twice, the difference of two radii squared instead of the difference of their squares, and $\frac{dr}{d\theta}$ read as the slope. These ten problems cover each of them.

- Parametric slope.** For $x = x(t)$, $y = y(t)$: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ where $\frac{dx}{dt} \neq 0$. The tangent is horizontal where $\frac{dy}{dt} = 0$ and vertical where $\frac{dx}{dt} = 0$ (with the other derivative nonzero).
- Second derivative.** $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{dx/dt}$: differentiate the slope in t , then divide by $\frac{dx}{dt}$ again.
- Arc length and area.** $L = \int_{\alpha}^{\beta} \sqrt{x'(t)^2 + y'(t)^2} dt$, the integral of the speed. The area under a curve traced once from left to right is $\int_{\alpha}^{\beta} y(t) x'(t) dt$.
- Polar coordinates.** $x = r \cos \theta$, $y = r \sin \theta$, $r^2 = x^2 + y^2$. A polar curve is the parametric curve $x = r(\theta) \cos \theta$, $y = r(\theta) \sin \theta$.
- Polar area and length.** The area swept as θ goes from α to β is $\frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta$, from thin wedges of area $\frac{1}{2} r^2 d\theta$. The length is $\int_{\alpha}^{\beta} \sqrt{r^2 + r'(\theta)^2} d\theta$. Between two curves, use $\frac{1}{2} \int (r_{\text{out}}^2 - r_{\text{in}}^2) d\theta$.

- **Polar slope.** $\frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}$, from the product rule on $r \sin \theta$ and $r \cos \theta$.

Builds on: Differentiation rules: chain, product, quotient, Arc length and surface area of revolution

Problems

1. For $x = t^2$, $y = t^3 - 3t$, find the slope and tangent line at $t = 2$, and the points where the tangent is horizontal or vertical. difficulty ·

2. For the same curve, find $\frac{d^2y}{dx^2}$ and its value at $t = 2$. difficulty ··

3. Identify the curve $x = 2 \cos t$, $y = 3 \sin t$, find its slope at $t = \frac{\pi}{4}$, and find the area it encloses. difficulty ·

4. Find the length of $x = t^2$, $y = \frac{2}{3}t^3$ for $0 \leq t \leq \sqrt{3}$. difficulty ··

5. A wheel of radius 1 rolls along the x -axis, and a point on its rim traces the cycloid $x = t - \sin t$, $y = 1 - \cos t$. Find the length of one arch, $0 \leq t \leq 2\pi$, and the area under it. difficulty ··

6. A point moves along $x = \sin^2 t$, $y = \cos^2 t$ for $0 \leq t \leq \pi$. How far does it travel, and how long is the curve it traces? difficulty ··

7. Show that $r = 2 \cos \theta$ is a circle, and find its area with a polar integral. difficulty ·

8. Find the area enclosed by the cardioid $r = 1 + \cos \theta$, and its length. difficulty ··

9. Find the area of the region inside $r = 3 \sin \theta$ and outside $r = 1 + \sin \theta$. difficulty ···

10. Find the slope of the cardioid $r = 1 + \cos \theta$ at $\theta = \frac{\pi}{2}$. difficulty ··

Answers

1. Slope $\frac{9}{4}$, tangent $y = \frac{9}{4}x - 7$; horizontal tangents at $(1, \pm 2)$, vertical at $(0, 0)$
2. $\frac{d^2y}{dx^2} = \frac{3(t^2 + 1)}{4t^3}$, which is $\frac{15}{32}$ at $t = 2$
3. An ellipse; slope $-\frac{3}{2}$ at $t = \frac{\pi}{4}$; area 6π
4. $\frac{2}{3}(8 - 1) = \frac{14}{3} \approx 4.667$
5. Length 8, area 3π
6. It travels $2\sqrt{2} \approx 2.828$; the segment it traces has length $\sqrt{2}$
7. π
8. Area $\frac{3\pi}{2} \approx 4.712$, length 8
9. π
10. $\frac{dy}{dx} = \frac{-1}{-1} = 1$

Worked solutions

Problem 1

For $x = t^2$, $y = t^3 - 3t$, find the slope and tangent line at $t = 2$, and the points where the tangent is horizontal or vertical.

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2t}$$

$\frac{dy}{dt}$ over $\frac{dx}{dt}$.

At $t = 2$: slope $\frac{12 - 3}{4} = \frac{9}{4}$, at the point $(4, 2)$

$$x = 4 \text{ and } y = 8 - 6 = 2.$$

Tangent: $y - 2 = \frac{9}{4}(x - 4)$, or $y = \frac{9}{4}x - 7$ Point-slope form.

Horizontal where $3t^2 - 3 = 0$, $t = \pm 1$: the points $(1, -2)$ and $(1, 2)$. Vertical where $2t = 0$, $t = 0$: the point $(0, 0)$

$$\text{At } t = 0, \frac{dy}{dt} = -3 \neq 0.$$

Slope $\frac{9}{4}$, tangent $y = \frac{9}{4}x - 7$;
horizontal tangents at $(1, \pm 2)$,
vertical at $(0, 0)$

The curve also crosses itself at $(3, 0)$, where $t = \pm\sqrt{3}$, with two different slopes, $\pm\sqrt{3}$.

Problem 2

For the same curve, find $\frac{d^2y}{dx^2}$ and its value at $t = 2$.

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2t} = \frac{3}{2}t - \frac{3}{2t}$$

Split the fraction before differentiating.

$$\frac{d}{dt} \left(\frac{dy}{dx} \right) = \frac{3}{2} + \frac{3}{2t^2}$$

The derivative of the slope with respect to t , not yet with respect to x .

$$\frac{d^2y}{dx^2} = \frac{\frac{3}{2} + \frac{3}{2t^2}}{2t} = \frac{3(t^2 + 1)}{4t^3}$$

Divide by $\frac{dx}{dt} = 2t$: the chain rule, $\frac{d}{dx} = \frac{1}{\frac{dx}{dt}} \frac{d}{dt}$.

$$\frac{d^2y}{dx^2} = \frac{3(t^2 + 1)}{4t^3}, \text{ which is } \frac{15}{32} \text{ at } t = 2$$

Positive, so the curve is concave up there. It is positive for every $t > 0$ and negative for every $t < 0$.

Problem 3

Identify the curve $x = 2 \cos t$, $y = 3 \sin t$, find its slope at $t = \frac{\pi}{4}$, and find the area it encloses.

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = \cos^2 t + \sin^2 t = 1$$

Eliminate t with the Pythagorean identity: an ellipse with semi-axes 2 and 3.

$$\frac{dy}{dx} = \frac{3 \cos t}{-2 \sin t}, \text{ which is } -\frac{3}{2} \text{ at } t = \frac{\pi}{4}, \text{ at the point } (\sqrt{2}, \frac{3\sqrt{2}}{2})$$

$$\cos \frac{\pi}{4} = \sin \frac{\pi}{4}.$$

$$\begin{aligned} \text{Area} &= \left| \int_0^{2\pi} y x'(t) dt \right| = \left| \int_0^{2\pi} 3 \sin t \cdot \right. \\ &\quad \left. (-2 \sin t) dt \right| = 6 \int_0^{2\pi} \sin^2 t dt \end{aligned}$$

The curve runs counterclockwise, from right to left along the top, so the integral comes out negative.

An ellipse; slope $-\frac{3}{2}$ at $t = \frac{\pi}{4}$; area 6π

$$\int_0^{2\pi} \sin^2 t dt = \pi. \text{ This is } \pi ab \text{ with } a = 2, b = 3.$$

Problem 4

Find the length of $x = t^2$, $y = \frac{2}{3}t^3$ for $0 \leq t \leq \sqrt{3}$.

$$\begin{aligned} x' &= 2t, y' = 2t^2, \text{ so } x'^2 + y'^2 = 4t^2 + \\ 4t^4 &= 4t^2(1 + t^2) \end{aligned}$$

Factor before taking the root.

$$\text{Speed} = 2t\sqrt{1 + t^2}$$

$$t \geq 0 \text{ here, so } \sqrt{4t^2} = 2t.$$

$$L = \int_0^{\sqrt{3}} 2t\sqrt{1 + t^2} dt = \frac{2}{3} \left[(1 + t^2)^{3/2} \right]_0^{\sqrt{3}}$$

Substitute $u = 1 + t^2$, $du = 2t dt$.

$$\frac{2}{3}(8 - 1) = \frac{14}{3} \approx 4.667$$

The curve runs from $(0, 0)$ to $(3, 2\sqrt{3})$, and the chord between them is $\sqrt{21} \approx 4.583$.

Problem 5

A wheel of radius 1 rolls along the x -axis, and a point on its rim traces the cycloid $x = t - \sin t$, $y = 1 - \cos t$. Find the length of one arch, $0 \leq t \leq 2\pi$, and the area under it.

$$x'^2 + y'^2 = (1 - \cos t)^2 + \sin^2 t = 2 - 2 \cos t = 4 \sin^2 \frac{t}{2}$$

The half-angle identity $1 - \cos t = 2 \sin^2 \frac{t}{2}$.

$$L = \int_0^{2\pi} 2 \left| \sin \frac{t}{2} \right| dt = \left[-4 \cos \frac{t}{2} \right]_0^{2\pi} = 4 + 4$$

$\sin \frac{t}{2} \geq 0$ on $[0, 2\pi]$, so the absolute value can go.

$$\text{Area} = \int_0^{2\pi} y x'(t) dt = \int_0^{2\pi} (1 - \cos t)^2 dt = \int_0^{2\pi} (1 - 2 \cos t + \cos^2 t) dt$$

x increases from 0 to 2π , left to right.

$$= 2\pi - 0 + \pi$$

$$\int_0^{2\pi} \cos^2 t dt = \pi.$$

Length 8, area 3π

The arch is 4 times as long as the wheel's diameter, and its area is 3 times the wheel's.

Problem 6

A point moves along $x = \sin^2 t$, $y = \cos^2 t$ for $0 \leq t \leq \pi$. How far does it travel, and how long is the curve it traces?

$$x + y = 1, \text{ with } 0 \leq x \leq 1$$

The point stays on the segment from $(0, 1)$ to $(1, 0)$.

$$x' = 2 \sin t \cos t = \sin 2t, y' = -\sin 2t, \text{ so the speed is } \sqrt{2} |\sin 2t|$$

$$\sqrt{2 \sin^2 2t} = \sqrt{2} |\sin 2t|, \text{ not } \sqrt{2} \sin 2t.$$

$$\text{Distance} = \sqrt{2} \int_0^\pi |\sin 2t| dt = \sqrt{2} \cdot 2$$

$\sin 2t$ is negative on $(\frac{\pi}{2}, \pi)$, so split there: each half contributes 1.

It travels $2\sqrt{2} \approx 2.828$; the segment it traces has length $\sqrt{2}$

At $t = 0$ it is at $(0, 1)$, at $t = \frac{\pi}{2}$ at $(1, 0)$, and at $t = \pi$ back at $(0, 1)$: the segment is covered twice. The arc length integral measures distance traveled, which is the length of the curve only if it is traced once.

Problem 7

Show that $r = 2 \cos \theta$ is a circle, and find its area with a polar integral.

$r^2 = 2r \cos \theta$, so $x^2 + y^2 = 2x$, which is $(x - 1)^2 + y^2 = 1$

Multiply by r , then convert. A circle of radius 1 centered at $(1, 0)$.

The circle is traced once as θ runs from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

$r = 0$ at both ends, and $r \geq 0$ in between.

Area $= \frac{1}{2} \int_{-\pi/2}^{\pi/2} 4 \cos^2 \theta \, d\theta = \int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta) \, d\theta$

$\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$.

π

The $\cos 2\theta$ part integrates to 0. A circle of radius 1 has area π , as it should.

Problem 8

Find the area enclosed by the cardioid $r = 1 + \cos \theta$, and its length.

Area $= \frac{1}{2} \int_0^{2\pi} (1 + \cos \theta)^2 \, d\theta = \frac{1}{2} \int_0^{2\pi} (1 + 2 \cos \theta + \cos^2 \theta) \, d\theta$

The cardioid is traced once as θ goes around once, and $r \geq 0$ throughout.

$= \frac{1}{2}(2\pi + 0 + \pi) = \frac{3\pi}{2}$

$\int_0^{2\pi} \cos^2 \theta \, d\theta = \pi$.

$r^2 + r'^2 = (1 + \cos \theta)^2 + \sin^2 \theta = 2 + 2 \cos \theta = 4 \cos^2 \frac{\theta}{2}$

The half-angle identity again.

$$L = \int_0^{2\pi} 2 \left| \cos \frac{\theta}{2} \right| d\theta = 2 \int_0^{\pi} 2 \cos \frac{\theta}{2} d\theta = 2 \left[4 \sin \frac{\theta}{2} \right]_0^{\pi}$$

$\cos \frac{\theta}{2} < 0$ for $\pi < \theta < 2\pi$, so use the symmetry about the x -axis and double the top half.

Area $\frac{3\pi}{2} \approx 4.712$, **length** 8

Without the absolute value the length integral would be $\left[4 \sin \frac{\theta}{2} \right]_0^{2\pi} = 0$.

Problem 9

Find the area of the region inside $r = 3 \sin \theta$ and outside $r = 1 + \sin \theta$.

$3 \sin \theta = 1 + \sin \theta$ gives $\sin \theta = \frac{1}{2}$, so $\theta = \frac{\pi}{6}$ and $\frac{5\pi}{6}$ Where the curves cross. Between them, $3 \sin \theta$ is the outer curve.

$$\text{Area} = \frac{1}{2} \int_{\pi/6}^{5\pi/6} (9 \sin^2 \theta - (1 + \sin \theta)^2) d\theta = \frac{1}{2} \int_{\pi/6}^{5\pi/6} (8 \sin^2 \theta - 2 \sin \theta - 1) d\theta$$

Outer radius squared minus inner radius squared.

$$\int_{\pi/6}^{5\pi/6} \sin^2 \theta d\theta = \frac{\pi}{3} + \frac{\sqrt{3}}{4},$$

$$\int_{\pi/6}^{5\pi/6} \sin \theta d\theta = \sqrt{3}, \quad \int_{\pi/6}^{5\pi/6} d\theta = \frac{2\pi}{3}$$

$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$ for the first.

$$\frac{1}{2} \left(\frac{8\pi}{3} + 2\sqrt{3} - 2\sqrt{3} - \frac{2\pi}{3} \right)$$

The $\sqrt{3}$ terms cancel.

π

The circle $r = 3 \sin \theta$ has area $\frac{9\pi}{4}$, so the region is a little under half of it.

Problem 10

Find the slope of the cardioid $r = 1 + \cos \theta$ at $\theta = \frac{\pi}{2}$.

$$r = 1 \text{ and } r' = -\sin \theta = -1 \text{ at } \theta = \frac{\pi}{2}$$

The point is $(x, y) = (0, 1)$.

$\frac{dy}{d\theta} = r' \sin \theta + r \cos \theta = -1 + 0 = -1$	The product rule on $y = r \sin \theta$.
$\frac{dx}{d\theta} = r' \cos \theta - r \sin \theta = 0 - 1 = -1$	The product rule on $x = r \cos \theta$.
$\frac{dy}{dx} = \frac{-1}{-1} = 1$	As θ increases past $\frac{\pi}{2}$ the point moves left and down, along a line of slope 1.

Where this goes wrong

1. Stopping at d/dt of dy/dx

The first derivative is a ratio of t -derivatives, so the second seems to be one more t -derivative.

$\frac{dy}{dx} = \frac{3}{2}t - \frac{3}{2t}$	Right so far: Problem 2, step 1.
“ $\frac{d^2y}{dx^2}$ is the derivative of this.”	The habit that causes the mistake: the derivative taken with respect to t , and treated as if it were with respect to x .
$\frac{d^2y}{dx^2} = \frac{3}{2} + \frac{3}{2t^2}$, which is $\frac{15}{8}$ at $t = 2$ ✗	That is $\frac{d}{dt}(\frac{dy}{dx})$. Changing to x needs one more division by $\frac{dx}{dt} = 2t$, and the answer is $\frac{15}{32}$, a quarter of this.

2. Dropping the absolute value in the speed

$\sqrt{a^2} = a$ looks like an ordinary simplification.

$x'^2 + y'^2 = 2 \sin^2 2t$ for $x = \sin^2 t$, $y = \cos^2 t$	Right so far: Problem 6, step 2.
“The speed is $\sqrt{2} \sin 2t$.”	The habit that causes the mistake: $\sqrt{a^2} = a $, and $\sin 2t$ is negative for half of the interval.
Distance = $\sqrt{2} \int_0^\pi \sin 2t \, dt = 0$ ✗	A point that moves the whole time cannot travel a distance of 0. The trip out cancels the trip back because the negative speed counts backward. With $ \sin 2t $, the distance is $2\sqrt{2}$.

3. Integrating over a curve traced twice

A full turn, 0 to 2π , is the default range for θ .

$$r = 2 \cos \theta \text{ and area} = \frac{1}{2} \int r^2 d\theta$$

Right so far: Problem 7.

“Integrate from 0 to 2π .”

The habit that causes the mistake: assuming every polar curve takes a full turn to trace once.

$$\text{Area} = \frac{1}{2} \int_0^{2\pi} 4 \cos^2 \theta d\theta = 2\pi \times$$

That is twice the area of a circle of radius 1. For $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$, r is negative and the same circle is traced a second time. Find the θ -range that traces the curve once, here $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, and the area is π .

4. Squaring the difference of the radii

The area between two curves in x is the integral of top minus bottom, so the polar version gets written the same way.

Outer $r = 3 \sin \theta$, inner $r = 1 + \sin \theta$, for
 $\frac{\pi}{6} \leq \theta \leq \frac{5\pi}{6}$

Right so far: Problem 9, step 1.

“Area = $\frac{1}{2} \int (r_{\text{out}} - r_{\text{in}})^2 d\theta$.”

The habit that causes the mistake: the difference taken before squaring.

$$\frac{1}{2} \int_{\pi/6}^{5\pi/6} (2 \sin \theta - 1)^2 d\theta = \pi - \frac{3\sqrt{3}}{2} \approx 0.544 \times$$

That is the area of wedges of length $r_{\text{out}} - r_{\text{in}}$ starting at the origin, not of the region between the curves. A wedge's area grows with the square of its radius, so subtract the two wedge areas:
 $\frac{1}{2} \int (r_{\text{out}}^2 - r_{\text{in}}^2) d\theta = \pi$.

5. Taking $dr/d\theta$ as the slope

For $y = f(x)$ the slope is the derivative, so for $r = f(\theta)$ it seems to be r' .

$r = 1 + \cos \theta$ at $\theta = \frac{\pi}{2}$, the point $(0, 1)$

Right so far: Problem 10, step 1.

“The slope is $\frac{dr}{d\theta}$.”

The habit that causes the mistake: r and θ treated like y and x .

Slope = $-\sin \frac{\pi}{2} = -1$ ✗

$\frac{dr}{d\theta}$ is how fast the distance from the origin changes, not the steepness of the curve. The slope needs x and y : $\frac{dy/d\theta}{dx/d\theta} = 1$, the opposite sign.

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