Newton's method for roots and minima

Ten problems on Newton's method: the update read off the tangent line, square roots and cube roots by iteration, a reciprocal computed without dividing, the exact error identity that makes the digits double, the slow convergence at a double root and its fix, a starting point that cycles, an iteration that runs away, Newton's method for a minimum, a system of two equations with its Jacobian, and a normal quantile found by inverting the distribution function, with worked solutions and the mistakes of adding the step instead of subtracting it, finding a root of the function when its minimum is wanted, expecting quadratic convergence at a double root, seeding the reciprocal iteration too high and transposing the Jacobian.

BEFORE YOU START

Newton's method finds a root of an equation $f(x) = 0$ by replacing f with its tangent line, solving the line instead, and repeating from the new point. Near a simple root it is remarkably fast, because the error at each step is roughly the square of the error before it, so the number of correct digits about doubles. It is how calculators take square roots, how hardware computes reciprocals, how statistical software inverts a distribution function, and, applied to f' instead of f, how second-order optimizers find a minimum. These ten problems derive the update, run it on square roots, cube roots and reciprocals, prove the doubling of digits, and then look at what goes wrong: a double root, a starting point that cycles, an iteration that runs away. They end with a minimum, a system of two equations and a normal quantile. The five mistakes at the end are the ones that still produce numbers: a step added instead of subtracted, the root of the function found when its minimum was wanted, quadratic convergence expected at a double root, a reciprocal iteration started too far out, and a Jacobian used transposed.

- **The update.** For a differentiable f and a current guess x_n with $f'(x_n) \neq 0$, Newton's method sets $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$.
- **Error.** If r is the root, $e_n = x_n - r$ is the error of the n-th guess. A root is **simple** when $f(r) = 0$ and $f'(r) \neq 0$, and has **multiplicity** m when $f(x) = (x - r)^m g(x)$ with $g(r) \neq 0$.
- **Quadratic convergence.** Near a simple root, with f'' continuous, $e_{n+1} \approx \dfrac{f''(r)}{2f'(r)}\, e_n^2$. Linear convergence means $e_{n+1} \approx c\, e_n$ with a constant $0 < |c| < 1$.
- **Minimization.** To find a minimum of f, apply the method to f': $x_{n+1} = x_n - \dfrac{f'(x_n)}{f''(x_n)}$. It finds a point with $f' = 0$, which is a minimum when $f'' > 0$ there.
- **Systems.** For $F : \mathbb{R}^n \to \mathbb{R}^n$ with Jacobian J, $J_{ij} = \partial F_i / \partial x_j$ (row i is equation i, column j is variable j), the step d solves $J(x_n)\, d = -F(x_n)$ and $x_{n+1} = x_n + d$.

- Φ is the standard normal distribution function and $\varphi(z) = e^{-z^2/2}/\sqrt{2\pi}$ its density, with $\Phi' = \varphi$.

Builds on: Taylor series and linear approximation

Problems

1. Derive the Newton update from the tangent line at x_n. Then apply it to $f(x) = x^2 - 2$ from $x_0 = 1$: show the update is $x_{n+1} = \frac{1}{2}\left(x_n + 2/x_n\right)$ and find x_1, x_2 and x_3 as fractions. difficulty ·

2. Write the Newton update for the cube root of $a > 0$, using $f(x) = x^3 - a$. Take $a = 10$ and $x_0 = 2$, and find x_1 and x_2. difficulty ·

3. To compute $1/a$ without dividing, apply Newton's method to $f(x) = \dfrac{1}{x} - a$. Show that the update is $x_{n+1} = x_n(2 - a x_n)$ and that $1 - a x_{n+1} = (1 - a x_n)^2$. Find x_1, x_2 and x_3 for $a = 7$ from $x_0 = 0.1$, and say for which x_0 the iteration converges. difficulty ··

4. For the square-root iteration of Problem 1 with $a = 2$, show that the errors satisfy $e_{n+1} = \dfrac{e_n^2}{2x_n}$ exactly. Compute e_0 to e_4 from $x_0 = 1$, and compare the limit of e_{n+1}/e_n^2 with $\dfrac{f''(r)}{2f'(r)}$. difficulty ··

5. $f(x) = x^3 - 3x + 2 = (x-1)^2(x+2)$ has a double root at $x = 1$. Starting from $x_0 = 2$, find x_1 and x_2 and the ratios e_{n+1}/e_n. Show that near 1 the ratio tends to $\frac{1}{2}$. Then run the modified step $x_{n+1} = x_n - 2f(x_n)/f'(x_n)$ from the same start. difficulty ··

6. Apply Newton's method to $f(x) = x^3 - 2x + 2$ from $x_0 = 0$, and then from $x_0 = -2$. What happens in each case? difficulty ··

7. Show that for $f(x) = x^{1/3}$ (the real cube root) the Newton update is $x_{n+1} = -2x_n$, whatever $x_0 \neq 0$. Then apply the method to $f(x) = \arctan x$ from $x_0 = 1$ and from $x_0 = 1.5$, and compute two steps of each. difficulty ···

8. Find the minimum of $f(x) = e^x - 2x$ with Newton's method from $x_0 = 0$: write the update, compute x_1 to x_3, and compare with the exact minimizer. difficulty ··

9. Solve the system $x^2 + y^2 = 4$, $xy = 1$ with Newton's method from $(x_0, y_0) = (2, 0.5)$: write F and its Jacobian, take one step, and compare with the exact solution near the start. difficulty ···

10. The 97.5% point of the standard normal is the root of $f(z) = \Phi(z) - 0.975$. With $\Phi(2) \approx 0.977250$ and $\varphi(2) \approx 0.053991$, take one Newton step from $z_0 = 2$. Why does the method suit this equation? difficulty ···

Answers

1. $x_1 = \frac{3}{2}$, $x_2 = \frac{17}{12} \approx 1.41667$, $x_3 = \frac{577}{408} \approx 1.4142157$
2. $x_{n+1} = \frac{1}{3}\left(2x_n + a/x_n^2\right)$; $x_1 = \frac{13}{6} \approx 2.16667$, $x_2 \approx 2.154504$
3. $x_{n+1} = x_n(2 - a x_n)$; $x_1 = 0.13$, $x_2 = 0.1417$, $x_3 = 0.14284777$; it converges exactly when $0 < x_0 < 2/a$
4. $e_{n+1} = e_n^2/(2x_n)$; the errors are about 0.0858, 0.00245, 2.1×10^{-6}, 1.6×10^{-12}; the ratio tends to $1/(2\sqrt{2})$, the general constant
5. Plain Newton: $x_1 = \frac{14}{9}$, $x_2 \approx 1.298$, ratio $\to \frac{1}{2}$ (linear); modified: errors 0.111, 0.0019, 6.3×10^{-7} (quadratic again)
6. From 0 the guesses cycle $0, 1, 0, 1, \ldots$ forever; from -2 they converge to the only real root, $r \approx -1.7692924$
7. $x^{1/3}$: $x_{n+1} = -2x_n$, diverging from every start; $\arctan$: from 1 the iterates go to 0, from 1.5 they alternate in sign and grow
8. $x_{n+1} = x_n - 1 + 2e^{-x_n}$; $x_1 = 1$, $x_2 \approx 0.7358$, $x_3 \approx 0.6940$; the minimizer is $\ln 2 \approx 0.6931$
9. $(x_1, y_1) = \left(\frac{29}{15}, \frac{31}{60}\right) \approx (1.93333, 0.51667)$; the solution is $\left(\frac{\sqrt{6}+\sqrt{2}}{2}, \frac{\sqrt{6}-\sqrt{2}}{2}\right) \approx (1.93185, 0.51764)$
10. $z_1 \approx 1.9583$; the iterates converge to 1.959964, the 1.96 of a 95% confidence interval

Worked solutions

Problem 1

> Derive the Newton update from the tangent line at x_n. Then apply it to $f(x) = x^2 - 2$ from $x_0 = 1$: show the update is $x_{n+1} = \frac{1}{2}\left(x_n + 2/x_n\right)$ and find x_1, x_2 and x_3 as fractions.

Tangent at x_n: $y = f(x_n) + f'(x_n)(x - x_n)$	The line through $(x_n, f(x_n))$ with slope $f'(x_n)$.
Set $y = 0$: $x = x_n - f(x_n)/f'(x_n)$	Solve for x, which needs $f'(x_n) \neq 0$. This is x_{n+1}.
$x_{n+1} = x_n - \dfrac{x_n^2 - 2}{2x_n}$	$f'(x) = 2x$.
$= \dfrac{2x_n^2 - x_n^2 + 2}{2x_n} = \dfrac{1}{2}\left(x_n + \dfrac{2}{x_n}\right)$	Common denominator, then split.
$x_1 = \frac{1}{2}(1 + 2) = \frac{3}{2}$, $x_2 = \frac{1}{2}\left(\frac{3}{2} + \frac{4}{3}\right) = \frac{17}{12}$	Each guess is the average of x_n and $2/x_n$.
$x_3 = \frac{1}{2}\left(\frac{17}{12} + \frac{24}{17}\right) = \frac{577}{408}$	Common denominator 204, then halve.
$x_1 = \frac{3}{2}$, $x_2 = \frac{17}{12} \approx 1.41667$, $x_3 = \frac{577}{408} \approx 1.4142157$	$\sqrt{2} \approx 1.4142136$, so x_3 is already right to six figures. If x_n is too big, $2/x_n$ is too small, and their average is closer than either: the Babylonian method for square roots.

Problem 2

> Write the Newton update for the cube root of $a > 0$, using $f(x) = x^3 - a$. Take $a = 10$ and $x_0 = 2$, and find x_1 and x_2.

$x_{n+1} = x_n - \dfrac{x_n^3 - a}{3x_n^2}$	$f'(x) = 3x^2$.
$= \dfrac{3x_n^3 - x_n^3 + a}{3x_n^2} = \dfrac{1}{3}\left(2x_n + \dfrac{a}{x_n^2}\right)$	Common denominator, then split.

$x_1 = \frac{1}{3}\left(4 + \frac{10}{4}\right) = \frac{13}{6} \approx 2.16667$	$a = 10$, $x_0 = 2$.
$x_2 = \frac{1}{3}\left(\frac{13}{3} + 10 \cdot \frac{36}{169}\right) \approx 2.154504$	$1/x_1^2 = 36/169$.
$x_{n+1} = \frac{1}{3}\left(2x_n + a/x_n^2\right)$; $x_1 = \frac{13}{6} \approx 2.16667$, $x_2 \approx 2.154504$	$\sqrt[3]{10} \approx 2.154435$, so the errors are about 0.0122 and then 0.000069. The next step, $x_3 \approx 2.1544347$, is off by about 2×10^{-9}.

Problem 3

> To compute $1/a$ without dividing, apply Newton's method to $f(x) = \dfrac{1}{x} - a$. Show that the update is $x_{n+1} = x_n(2 - a x_n)$ and that $1 - a x_{n+1} = (1 - a x_n)^2$. Find x_1, x_2 and x_3 for $a = 7$ from $x_0 = 0.1$, and say for which x_0 the iteration converges.

$f'(x) = -1/x^2$	Power rule.
$x_{n+1} = x_n - \dfrac{1/x_n - a}{-1/x_n^2}$	The update.
$= x_n + x_n^2\left(1/x_n - a\right) = x_n(2 - a x_n)$	Dividing by $-1/x_n^2$ is multiplying by $-x_n^2$. Only multiplications and a subtraction are left.
$1 - a x_{n+1} = 1 - 2a x_n + a^2 x_n^2 = (1 - a x_n)^2$	Expand $a x_n(2 - a x_n)$.
$x_1 = 0.1(2 - 0.7) = 0.13$, $x_2 = 0.13(2 - 0.91) = 0.1417$	$a = 7$.
$x_3 = 0.1417(2 - 0.9919) = 0.14284777$	Exact in decimals.
$x_{n+1} = x_n(2 - a x_n)$; $x_1 = 0.13$, $x_2 = 0.1417$, $x_3 = 0.14284777$; **it converges exactly when** $0 < x_0 < 2/a$	With $\varepsilon_n = 1 - a x_n$, step 4 gives $\varepsilon_n = \varepsilon_0^{2^n}$: here 0.3, 0.09, 0.0081, 0.0000656, against $1/7 \approx 0.14285714$. That tends to 0 exactly when $\lvert\varepsilon_0\rvert < 1$, that is $0 < a x_0 < 2$ (Mistake 4).

Problem 4

For the square-root iteration of Problem 1 with $a = 2$, show that the errors satisfy $e_{n+1} = \dfrac{e_n^2}{2x_n}$ exactly. Compute e_0 to e_4 from $x_0 = 1$, and compare the limit of e_{n+1}/e_n^2 with $\dfrac{f''(r)}{2f'(r)}$.

$x_{n+1} - \sqrt{2} = \dfrac{x_n^2 + 2}{2x_n} - \sqrt{2}$	Problem 1's update, written over one denominator.
$= \dfrac{x_n^2 - 2\sqrt{2}\,x_n + 2}{2x_n} = \dfrac{(x_n - \sqrt{2})^2}{2x_n}$	Common denominator; the numerator is a perfect square.
$e_0 \approx -0.414$, $e_1 \approx 0.0858$, $e_2 \approx 0.00245$	$x_0 = 1$, $x_1 = 1.5$, $x_2 = 17/12$.
$e_3 \approx 2.12 \times 10^{-6}$, $e_4 \approx 1.59 \times 10^{-12}$	$x_3 = 577/408$, and $x_4 = 665857/470832$.
$e_{n+1}/e_n^2 = 1/(2x_n) \to 1/(2\sqrt{2}) \approx 0.354$	Step 2, with $x_n \to \sqrt{2}$.
$\dfrac{f''(r)}{2f'(r)} = \dfrac{2}{2 \cdot 2\sqrt{2}} = \dfrac{1}{2\sqrt{2}}$	$f'' = 2$ and $f'(\sqrt{2}) = 2\sqrt{2}$.
$e_{n+1} = e_n^2/(2x_n)$; **the errors are about** 0.0858, 0.00245, 2.1×10^{-6}, 1.6×10^{-12}; **the ratio tends to** $1/(2\sqrt{2})$, **the general constant**	The number of correct digits goes about $1, 2, 5, 11$, roughly doubling each step. Every error after the first is positive, so from x_1 on the guesses come down to $\sqrt{2}$ from above.

Problem 5

$f(x) = x^3 - 3x + 2 = (x-1)^2(x+2)$ has a double root at $x = 1$. Starting from $x_0 = 2$, find x_1 and x_2 and the ratios e_{n+1}/e_n. Show that near 1 the ratio tends to $\frac{1}{2}$. Then run the modified step $x_{n+1} = x_n - 2f(x_n)/f'(x_n)$ from the same start.

$f'(x) = 3x^2 - 3$; $x_1 = 2 - \frac{4}{9} = \frac{14}{9} \approx 1.5556$	$f(2) = 4$, $f'(2) = 9$.
$x_2 \approx 1.2979$; errors 1, 0.556, 0.298	One more step; the ratios are 0.556 and 0.536.

$\dfrac{f}{f'} = \dfrac{(x-1)^2(x+2)}{(x-1)(3x+3)} = \dfrac{(x-1)(x+2)}{3(x+1)}$	$f' = 3(x-1)(x+1)$; one factor $x-1$ cancels.
$e_{n+1} = e_n\left(1 - \dfrac{x_n+2}{3(x_n+1)}\right) \to e_n\left(1 - \frac{3}{6}\right)$	Subtract step 3 from $x_n - 1$; at $x_n = 1$ the bracket is $\frac{1}{2}$.
Modified: $x_1 = 2 - \frac{8}{9} = \frac{10}{9}$, then $x_2 \approx 1.00195$, $x_3 \approx 1.00000063$	Twice the Newton step from each point.
Plain Newton: $x_1 = \frac{14}{9}$, $x_2 \approx 1.298$, **ratio** $\to \frac{1}{2}$ **(linear); modified: errors** 0.111, 0.0019, 6.3×10^{-7} **(quadratic again)**	At a root of multiplicity m, $f/f' \approx e/m$, so a plain step removes only $1/m$ of the error and the ratio is $(m-1)/m$. Multiplying the step by m restores the fast convergence, when m is known.

Problem 6

> Apply Newton's method to $f(x) = x^3 - 2x + 2$ from $x_0 = 0$, and then from $x_0 = -2$. What happens in each case?

$f'(x) = 3x^2 - 2$, so $x_{n+1} = x_n - \dfrac{x_n^3 - 2x_n + 2}{3x_n^2 - 2}$	The update.
$x_1 = 0 - \frac{2}{-2} = 1$	$f(0) = 2$, $f'(0) = -2$.
$x_2 = 1 - \frac{1}{1} = 0$	$f(1) = 1$, $f'(1) = 1$.
From -2: $x_1 = -2 - \frac{-2}{10} = -1.8$, then $x_2 \approx -1.76995$	$f(-2) = -2$, $f'(-2) = 10$.
$x_3 \approx -1.769293$, $x_4 \approx -1.7692924$	Two more steps; the iterates stop changing in the seventh figure.

From 0 the guesses cycle $0, 1, 0, 1, \ldots$ forever; from -2 they converge to the only real root, $r \approx -1.7692924$	The tangent at 0 crosses the axis at 1, and the tangent at 1 crosses it at 0. Newton's method only promises convergence from a start close enough to a root. Here f has a local maximum and a local minimum at $x = \pm\sqrt{2/3} \approx \pm 0.816$, where $f' = 0$ and the update is not even defined, and the starts 0 and 1 sit on either side of the local minimum, far from the root.

Problem 7

> Show that for $f(x) = x^{1/3}$ (the real cube root) the Newton update is $x_{n+1} = -2x_n$, whatever $x_0 \neq 0$. Then apply the method to $f(x) = \arctan x$ from $x_0 = 1$ and from $x_0 = 1.5$, and compute two steps of each.

$f'(x) = \frac{1}{3}x^{-2/3}$ for $x \neq 0$	Power rule; it holds for negative x too, with the real cube root.
$\dfrac{f(x)}{f'(x)} = \dfrac{x^{1/3}}{\frac{1}{3}x^{-2/3}} = 3x$	$x^{1/3} \cdot x^{2/3} = x$.
$x_{n+1} = x_n - 3x_n = -2x_n$	So from $x_0 = 1$: -2, 4, -8, …, doubling away from the root 0.
For arctan: $x_{n+1} = x_n - (1 + x_n^2)\arctan x_n$	$f'(x) = 1/(1+x^2)$.
From 1: $x_1 = 1 - 2 \cdot \frac{\pi}{4} \approx -0.5708$, $x_2 \approx 0.1169$	Then $x_3 \approx -0.0011$, closing in on 0.
From 1.5: $x_1 \approx 1.5 - 3.25 \cdot 0.98279 \approx -1.6941$, $x_2 \approx 2.3211$	$\arctan 1.5 \approx 0.98279$; then $x_3 \approx -5.114$.

$x^{1/3}$: $x_{n+1} = -2x_n$, **diverging from every start;** arctan: **from** 1 **the iterates go to** 0, **from** 1.5 **they alternate in sign and grow**	At the root of $x^{1/3}$ the derivative is infinite, so the theory's assumption $f'(r) \neq 0$ finite fails, and f/f' is three times the distance to the root. arctan has a perfectly good simple root, but it flattens out, so a tangent from far away overshoots; the starts that converge are those with $\lvert x_0 \rvert < 1.3917$ about.

Problem 8

> Find the minimum of $f(x) = e^x - 2x$ with Newton's method from $x_0 = 0$: write the update, compute x_1 to x_3, and compare with the exact minimizer.

$f'(x) = e^x - 2$, $f''(x) = e^x$	The step uses the first two derivatives.
$x_{n+1} = x_n - \dfrac{e^{x_n} - 2}{e^{x_n}} = x_n - 1 + 2e^{-x_n}$	Newton's method on f'.
$x_1 = 0 - 1 + 2 = 1$	$e^0 = 1$.
$x_2 = 1 - 1 + 2/e \approx 0.735759$	$2e^{-1}$.
$x_3 = x_2 - 1 + 2e^{-x_2} \approx 0.694042$	$e^{-0.735759} \approx 0.479142$.
Exact: $e^x = 2$, so $x^* = \ln 2 \approx 0.693147$	$f'' = e^x > 0$ everywhere, so this is the global minimum.
$x_{n+1} = x_n - 1 + 2e^{-x_n}$; $x_1 = 1$, $x_2 \approx 0.7358$, $x_3 \approx 0.6940$; **the minimizer is** $\ln 2 \approx 0.6931$	The errors are 0.307, 0.043, 0.0009 and then 4×10^{-7}: quadratic again, now for f'. Each step jumps to the minimum of the quadratic Taylor model at x_n. The minimum value is $2 - 2\ln 2 \approx 0.614 > 0$, so f itself has no root (Mistake 2).

Problem 9

Solve the system $x^2 + y^2 = 4$, $xy = 1$ with Newton's method from $(x_0, y_0) = (2, 0.5)$: write F and its Jacobian, take one step, and compare with the exact solution near the start.

$F(x, y) = \big(x^2 + y^2 - 4,\ xy - 1\big)$	Both equations moved to the form $F = 0$.
$J = \begin{bmatrix} 2x & 2y \\ y & x \end{bmatrix}$	Row i holds the partial derivatives of F_i in x and then y.
$F(2, 0.5) = (0.25, 0)$, $J = \begin{bmatrix} 4 & 1 \\ 0.5 & 2 \end{bmatrix}$	At the start.
$4d_1 + d_2 = -0.25$ and $0.5d_1 + 2d_2 = 0$	$J\,d = -F$.
$d_1 = -4d_2$, so $-15d_2 = -0.25$	The second equation, substituted into the first.
$d_2 = \frac{1}{60}$, $d_1 = -\frac{1}{15}$	So $(x_1, y_1) = (2 - \frac{1}{15},\ 0.5 + \frac{1}{60})$.
Exact: $(x + y)^2 = 6$, $(x - y)^2 = 2$	$x^2 + y^2 \pm 2xy = 4 \pm 2$.
$(x_1, y_1) = \left(\frac{29}{15}, \frac{31}{60}\right) \approx (1.93333, 0.51667)$; **the solution is** $\left(\frac{\sqrt{6}+\sqrt{2}}{2}, \frac{\sqrt{6}-\sqrt{2}}{2}\right) \approx (1.93185, 0.51764)$	The errors fall from about 0.07 to 0.0015 in one step and to 1.1×10^{-6} in the next. The step solves the linearized system, so a 2×2 solve takes the place of the division by f'.

Problem 10

The 97.5% point of the standard normal is the root of $f(z) = \Phi(z) - 0.975$. With $\Phi(2) \approx 0.977250$ and $\varphi(2) \approx 0.053991$, take one Newton step from $z_0 = 2$. Why does the method suit this equation?

$f'(z) = \varphi(z)$	$\Phi' = \varphi$, and the constant drops out.
$z_1 = 2 - \dfrac{0.002250}{0.053991}$	$f(2) = 0.977250 - 0.975$.

$z_1 \approx 2 - 0.04167 = 1.95833$	The step is negative because $\Phi(2)$ is already above 0.975.
$z_2 \approx 1.959961$, $z_3 \approx 1.9599640$	Repeat with Φ and φ at the new point.
$z_1 \approx 1.9583$; **the iterates converge to** 1.959964, **the** 1.96 **of a** 95% **confidence interval**	The derivative of a distribution function is its density, which is cheap to evaluate and positive, so every step is defined, and the root is simple. The errors are about 0.0016, then 2.6×10^{-6}, then about 7×10^{-12}: the digits double each step.

Where this goes wrong

1. Adding the step instead of subtracting it

The update is remembered as "x plus a correction", and the sign of the correction is easy to flip.

$f(x) = x^2 - 2$, $x_0 = 1$, $f(1) = -1$, $f'(1) = 2$	Right so far: Problem 1.
"$x_1 = x_0 + f(x_0)/f'(x_0)$."	The tangent line crosses zero at $x_0 - f(x_0)/f'(x_0)$: below the axis with a positive slope, the crossing is to the right.
$x_1 = 1 + \frac{-1}{2} = 0.5$ ✗	The guess moves away from $\sqrt{2}$, and the next wrong step jumps to -1.25, on the far side of 0. The right step gives $x_1 = 1 - \frac{-1}{2} = 1.5$. A quick check: $\lvert f(x_1)\rvert$ should be smaller than $\lvert f(x_0)\rvert$, and $f(0.5) = -1.75$.

2. Finding a root of f when its minimum is wanted

The same formula finds roots and minima, and it is easy to feed it the function instead of its derivative.

Minimize $f(x) = e^x - 2x$ from $x_0 = 0$	Right so far: Problem 8.

“Newton's step is $x - f/f'$.”	That finds where $f = 0$. A minimum is where $f' = 0$, so the step is $x - f'/f''$.
$x_1 = 0 - \frac{1}{-1} = 1$, $x_2 = 1 - \frac{e-2}{e-2} = 0$ ✗	The first step happens to agree with Problem 8, which hides the slip; then the guesses cycle $0, 1, 0, 1$. They cannot settle anywhere, because $f \geq 2 - 2\ln 2 \approx 0.614$ has no root. The minimizer is $\ln 2 \approx 0.693$.

3. Expecting quadratic convergence at a double root

Newton's method is famous for doubling the digits, and the promise is quoted without its condition $f'(r) \neq 0$.

$f(x) = (x-1)^2(x+2)$, $x_0 = 2$	Right so far: Problem 5.
“Ten steps of Newton's method are plenty for ten digits.”	That holds at a simple root. Here $f'(1) = 0$, and each step only halves the error.
After ten steps, x_{10} is 1 to about ten digits ✗	It is about 1.0013: the error is 1.3×10^{-3}, and twenty steps reach only about 1.2×10^{-6}. The ratio $e_{n+1}/e_n \to \frac{1}{2}$ is the sign of a double root; the step $x - 2f/f'$ of Problem 5 gets six digits in three steps.

4. Seeding the reciprocal iteration too high

Any rough guess near $1/a$ looks good enough, and 0.3 seems a fair guess for $1/7$.

$x_{n+1} = x_n(2 - 7x_n)$	Right so far: Problem 3 with $a = 7$.
“Start from $x_0 = 0.3$.”	Convergence needs $\lvert 1 - 7x_0 \rvert < 1$, that is $0 < x_0 < 2/7 \approx 0.2857$; here $1 - 7x_0 = -1.1$.

$x_1 = 0.3(2 - 2.1) = -0.03$, $x_2 = -0.03(2 + 0.21) = -0.0663$ ✗	The quantity $1 - 7x_n$ is squared at each step, from -1.1 to 1.21, then 1.4641, and the guesses run off to $-\infty$. Hardware first reads a guess from a table, which keeps $\lvert 1 - ax_0 \rvert$ well below 1.

5. Using the Jacobian transposed

In two variables the step needs a matrix, and the rows and columns of J are easy to swap.

$F(2, 0.5) = (0.25, 0)$	Right so far: Problem 9, step 3.
"$J = \begin{bmatrix} 2x & y \\ 2y & x \end{bmatrix}$, with each column one equation."	Row i of the Jacobian is equation i: the first row is $(2x, 2y)$.
$(x_1, y_1) \approx (1.93333, 0.53333)$ ✗	The transposed system gives $d_2 = \frac{1}{30}$ instead of $\frac{1}{60}$, so y overshoots: its error only shrinks from 0.018 to 0.016, where the right step brings it to 0.001, and the first equation is off by 0.022 instead of 0.005. Here the transposed steps still converge, but only linearly: after five of them the error is about 2×10^{-5}, where Newton's own steps reach 8×10^{-13} in three.

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