Line integrals and Green's theorem: work, potentials and circulation

Ten problems on line integrals and Green's theorem: the mass of a semicircular wire with ds = |r′(t)| dt, the work of a force along a parabola, one field integrated along two paths with different answers, the cross-partials test and a potential function, the fundamental theorem for line integrals on an awkward curve, the circulation around the unit circle by Green's theorem, the area of an ellipse from ½ ∮(x dy − y dx), outward flux and the divergence, the field around a hole that circulates 2π, and gradient descent against the rotating field of a two-player game, with worked solutions and the mistakes of dropping |r′(t)|, a clockwise boundary, the curl written backward, a zero curl read as a conservative field and dx = dt with dy left alone.

BEFORE YOU START

A line integral adds something up along a curve instead of across an interval. With respect to arc length it totals a density along a wire; with a vector field it totals the work a force does on an object moving along the path. Every one is computed the same way: parametrize the curve, rewrite everything in the parameter t, and do an ordinary integral. Two shortcuts save most of the work. If the field is a gradient, $F = \nabla f$, the work depends only on the endpoints, and it is $f(\text{end}) - f(\text{start})$. If the curve is closed, Green's theorem trades the line integral for a double integral over the region inside:

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

The same ideas explain why gradient descent can never go in circles while the simultaneous updates of a two-player game can. These ten problems compute line integrals directly, test fields for being conservative, use potentials and Green's theorem, and end with that contrast.

- A curve C is given by $r(t) = (x(t), y(t))$ for $a \le t \le b$, traced from $r(a)$ to $r(b)$. Its speed is $\|r'(t)\| = \sqrt{x'(t)^2 + y'(t)^2}$ and its arc-length element is $ds = \|r'(t)\|\,dt$.
- **Scalar line integral.** $\int_C g\,ds = \int_a^b g(r(t))\,\|r'(t)\|\,dt$. It does not depend on the direction of travel.
- **Work integral.** For $F = (P, Q)$, $\int_C F \cdot dr = \int_C P\,dx + Q\,dy = \int_a^b F(r(t)) \cdot r'(t)\,dt$, with $dx = x'(t)\,dt$ and $dy = y'(t)\,dt$. Reversing the direction changes its sign.
- **Conservative fields.** F is conservative if $F = \nabla f$ for some potential f. Then $\int_C F \cdot dr = f(r(b)) - f(r(a))$ for every path, and the integral around every closed curve is 0. On a region with no holes, F is conservative exactly when $\partial P/\partial y = \partial Q/\partial x$.

- **Green's theorem.** For a closed curve C traced counterclockwise (the region R on the left) and P, Q with continuous partial derivatives on all of R: $\oint_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA$. Taking $P = -\frac{y}{2}$, $Q = \frac{x}{2}$ gives $\text{Area}(R) = \frac{1}{2}\oint_C x\,dy - y\,dx$.
- **Flux form.** With n the outward unit normal, $\oint_C F \cdot n\,ds = \oint_C P\,dy - Q\,dx = \iint_R (P_x + Q_y)\,dA$, the integral of the divergence.

Builds on: Parametric and polar curves: slopes, arc length and area, Double integrals: iterated integrals, switching the order and polar coordinates, Partial derivatives, gradients and Hessians

Problems

1. A wire lies along the upper half of the circle $x^2 + y^2 = 9$, and its density is $\rho(x, y) = y$. Find the mass of the wire. difficulty ·

2. Find the work done by $F(x, y) = (x^2, xy)$ on an object moving along the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$. difficulty ·

3. Let $F(x, y) = (y, -x)$. Compute $\int_C F \cdot dr$ from $(0, 0)$ to $(1, 1)$ along the line $y = x$ and along the parabola $y = x^2$. What does the difference say about F? Check the difference with Green's theorem. difficulty ··

4. Show that $F(x, y) = (2xy + 3,\ x^2 + 4y)$ is conservative and find a potential f with $\nabla f = F$. difficulty ··

5. For F of Problem 4, compute $\int_C F \cdot dr$ along $r(t) = (1 + t^2, t^3)$, $0 \le t \le 1$, both with the potential and directly. Then give the integral of F around any closed curve. difficulty ··

6. Use Green's theorem to compute $\oint_C (x - y^3)\,dx + (x^3 + y)\,dy$, where C is the unit circle traced counterclockwise. Check it directly. difficulty ··

7. Use $\text{Area} = \frac{1}{2}\oint_C x\,dy - y\,dx$ to find the area inside the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$. difficulty ·

8. Find the outward flux of $F(x, y) = (x + y^2,\ 2y + x)$ across the circle $x^2 + y^2 = 4$, by the divergence form of Green's theorem and directly. difficulty ··

9. Let $F(x, y) = \left(\dfrac{-y}{x^2 + y^2},\ \dfrac{x}{x^2 + y^2} \right)$, defined everywhere except the origin. Show that $Q_x = P_y$, compute $\oint F \cdot dr$ counterclockwise around the unit circle, and find it around the circle of radius 1 centered at $(3, 0)$ and around the ellipse of Problem 7. difficulty ···

10. Gradient descent on $L(w_1, w_2) = \tfrac{1}{2}(w_1^2 + 4w_2^2)$ moves along the field $V = -\nabla L$. A two-player game where one player minimizes and the other maximizes $w_1 w_2$ moves along $G = (-w_2, w_1)$. Compute $\int V \cdot dr$ along the segment from $(2, 1)$ to $(0, 0)$ and compare it with L. Then compute the circulation of V and of G counterclockwise around the circle of radius 2 about the origin, and say what each means for the dynamics. difficulty ···

Answers

1. Mass $= 18$
2. $W = \frac{11}{15} \approx 0.7333$
3. 0 along the line, $-\frac{1}{3}$ along the parabola: F is not conservative
4. $f(x, y) = x^2 y + 3x + 2y^2$ (plus any constant)
5. $\int_C F \cdot dr = 9$; around any closed curve it is 0
6. $\oint_C = \frac{3\pi}{2} \approx 4.712$
7. Area $= 6\pi \approx 18.85$
8. Outward flux $= 12\pi \approx 37.70$
9. $\oint = 2\pi$ around the unit circle, 0 around the circle centered at $(3, 0)$, and 2π around the ellipse
10. $\int V \cdot dr = 4 = L(2, 1) - L(0, 0)$; circulation 0 for V and $8\pi \approx 25.13$ for G

Worked solutions

Problem 1

> A wire lies along the upper half of the circle $x^2 + y^2 = 9$, and its density is $\rho(x, y) = y$. Find the mass of the wire.

$r(t) = (3\cos t, 3\sin t)$, $0 \le t \le \pi$	The upper half of the circle of radius 3.
$r'(t) = (-3\sin t, 3\cos t)$ and $\lVert r'(t) \rVert = 3$	The speed is the radius, so $ds = 3\,dt$.
$\int_C y\,ds = \int_0^{\pi} 3\sin t \cdot 3\,dt$	Substitute $y = 3\sin t$ and ds.
$= 9\big[-\cos t\big]_0^{\pi} = 9 \cdot 2$	$-\cos\pi + \cos 0 = 2$.
Mass $= 18$	The wire is heaviest at the top, where $y = 3$. Leaving out $\lVert r'(t) \rVert$ gives 6, a third of the mass (Mistake 1).

Problem 2

> Find the work done by $F(x, y) = (x^2, xy)$ on an object moving along the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$.

$r(t) = (t, t^2)$, $0 \le t \le 1$, so $dx = dt$ and $dy = 2t\,dt$	Use x itself as the parameter.
$P = x^2 = t^2$ and $Q = xy = t \cdot t^2 = t^3$	Everything in terms of t.
$\int_C P\,dx + Q\,dy = \int_0^1 \left(t^2 + t^3 \cdot 2t\right) dt = \int_0^1 (t^2 + 2t^4)\,dt$	$F \cdot r'(t)$ with $r'(t) = (1, 2t)$.
$= \frac{1}{3} + \frac{2}{5}$	Integrate each power.
$W = \frac{11}{15} \approx 0.7333$	Writing dy as dt too gives $\frac{7}{12}$ (Mistake 5).

Problem 3

> Let $F(x, y) = (y, -x)$. Compute $\int_C F \cdot dr$ from $(0, 0)$ to $(1, 1)$ along the line $y = x$ and along the parabola $y = x^2$. What does the difference say about F? Check the difference with Green's theorem.

Line: $r(t) = (t, t)$, $r' = (1, 1)$, $F \cdot r' = t - t = 0$, so the integral is 0	The field is perpendicular to the line at every point.
Parabola: $r(t) = (t, t^2)$, $r' = (1, 2t)$, $F \cdot r' = t^2 - t \cdot 2t = -t^2$	$P = y = t^2$ and $Q = -x = -t$.
$\int_0^1 -t^2 \, dt = -\frac{1}{3}$	The two paths give different work.
Out along the parabola and back along the line is a closed loop around the region $x^2 \le y \le x$, traced counterclockwise, with $\oint = -\frac{1}{3} - 0 = -\frac{1}{3}$	The parabola runs below the line, so the region is on the left.
0 along the line, $-\frac{1}{3}$ along the parabola: F is not conservative	Green: $Q_x - P_y = -1 - 1 = -2$, and the region has area $\int_0^1 (x - x^2) \, dx = \frac{1}{6}$, so $\iint_R -2 \, dA = -\frac{1}{3}$, as found.

Problem 4

> Show that $F(x, y) = (2xy + 3,\ x^2 + 4y)$ is conservative and find a potential f with $\nabla f = F$.

$P_y = 2x$ and $Q_x = 2x$	Equal on the whole plane, which has no holes, so F is conservative.
$f = \int (2xy + 3) \, dx = x^2 y + 3x + g(y)$	Integrate P in x; the constant of integration may depend on y.
$f_y = x^2 + g'(y) = x^2 + 4y$, so $g'(y) = 4y$	Match f_y to Q. The x's must cancel here, and they do because of step 1.
$g(y) = 2y^2$	Any constant can be added.

$f(x,y) = x^2y + 3x + 2y^2$ **(plus any constant)**	Check: $f_x = 2xy + 3$ and $f_y = x^2 + 4y$.

Problem 5

> For F of Problem 4, compute $\int_C F \cdot dr$ along $r(t) = (1 + t^2, t^3)$, $0 \le t \le 1$, both with the potential and directly. Then give the integral of F around any closed curve.

$r(0) = (1,0)$ and $r(1) = (2,1)$	Only the endpoints matter for a gradient field.
$f(2,1) - f(1,0) = (4 + 6 + 2) - (0 + 3 + 0) = 12 - 3 = 9$	The fundamental theorem for line integrals, with $f = x^2y + 3x + 2y^2$.
Directly: $r' = (2t, 3t^2)$, $P = 2(1 + t^2)t^3 + 3$, $Q = (1 + t^2)^2 + 4t^3$	The long way, to confirm.
$\int_0^1 \left[(2t^3 + 2t^5 + 3)\,2t + (1 + 2t^2 + t^4 + 4t^3)\,3t^2\right] dt = \int_0^1 (6t + 3t^2 + 4t^4 + 6t^4 + 12t^5 + 4t^6 + 3t^6)\,dt$	Expand $F \cdot r'$ and collect: $6t + 3t^2 + 10t^4 + 12t^5 + 7t^6$.
$\int_C F \cdot dr = 9$; **around any closed curve it is** 0	Step 4 is $3 + 1 + 2 + 2 + 1 = 9$. A closed curve ends where it starts, so $f(\text{end}) - f(\text{start}) = 0$.

Problem 6

> Use Green's theorem to compute $\oint_C (x - y^3)\,dx + (x^3 + y)\,dy$, where C is the unit circle traced counterclockwise. Check it directly.

$P = x - y^3$, $Q = x^3 + y$, so $Q_x - P_y = 3x^2 - (-3y^2) = 3x^2 + 3y^2$	Differentiate Q in x, then subtract P differentiated in y.
$\iint_R 3(x^2 + y^2)\,dA = \int_0^{2\pi}\int_0^1 3r^2 \cdot r\,dr\,d\theta$	Polar coordinates over the unit disk, with the r in $r\,dr\,d\theta$.

$= 2\pi \cdot \frac{3}{4}$	$\int_0^1 3r^3\,dr = \frac{3}{4}$.
Directly: $x = \cos t$, $y = \sin t$, and $\int_0^{2\pi} \left[(\cos t - \sin^3 t)(-\sin t) + (\cos^3 t + \sin t)\cos t\right] dt = \int_0^{2\pi} (\sin^4 t + \cos^4 t)\,dt$	The terms $-\sin t \cos t$ and $\sin t \cos t$ cancel; each fourth power integrates to $\frac{3\pi}{4}$.
$\oint_C = \frac{3\pi}{2} \approx 4.712$	Writing the integrand as $P_y - Q_x$ gives $-\frac{3\pi}{2}$ (Mistake 3).

Problem 7

> Use Area $= \frac{1}{2}\oint_C x\,dy - y\,dx$ to find the area inside the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$.

$x = 3\cos t$, $y = 2\sin t$, $0 \le t \le 2\pi$	Counterclockwise, starting at $(3, 0)$.
$dx = -3\sin t\,dt$ and $dy = 2\cos t\,dt$	Both differentials in t.
$x\,dy - y\,dx = (6\cos^2 t + 6\sin^2 t)\,dt = 6\,dt$	$\cos^2 + \sin^2 = 1$.
$\frac{1}{2}\int_0^{2\pi} 6\,dt = 6\pi$	A constant integrand.
Area $= 6\pi \approx 18.85$	In general πab. Tracing the ellipse clockwise gives -6π (Mistake 2).

Problem 8

> Find the outward flux of $F(x, y) = (x + y^2,\ 2y + x)$ across the circle $x^2 + y^2 = 4$, by the divergence form of Green's theorem and directly.

$\operatorname{div} F = P_x + Q_y = 1 + 2 = 3$	The y^2 and x terms do not survive.
$\iint_R 3\,dA = 3 \cdot \pi \cdot 2^2 = 12\pi$	A constant divergence times the area of the disk of radius 2.

Directly: $r = (2\cos t, 2\sin t)$, $n = (\cos t, \sin t)$ and $ds = 2\,dt$	On a circle the outward normal points along the radius.
$F \cdot n = (2\cos t + 4\sin^2 t)\cos t + (4\sin t + 2\cos t)\sin t = 2\cos^2 t + 4\sin^2 t + 4\sin^2 t\cos t + 2\sin t\cos t$	The last two terms integrate to 0 over a full turn, and the first two to $2\pi + 4\pi = 6\pi$.
Outward flux $= 12\pi \approx 37.70$	6π times the 2 from $ds = 2\,dt$.

Problem 9

Let $F(x, y) = \left(\frac{-y}{x^2 + y^2},\ \frac{x}{x^2 + y^2}\right)$, defined everywhere except the origin. Show that $Q_x = P_y$, compute $\oint F \cdot dr$ counterclockwise around the unit circle, and find it around the circle of radius 1 centered at $(3, 0)$ and around the ellipse of Problem 7.

$Q_x = \frac{(x^2 + y^2) - 2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$ and $P_y = -\frac{(x^2 + y^2) - 2y^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$	Quotient rule on each.
Unit circle: $F = (-\sin t, \cos t)$ and $r' = (-\sin t, \cos t)$, so $F \cdot r' = 1$ and $\oint = 2\pi$	On the unit circle $x^2 + y^2 = 1$.
Green's theorem does not apply to the unit disk: F is undefined at the origin, inside it	That is how a field with $Q_x - P_y = 0$ can still circulate.
The circle centered at $(3, 0)$ does not enclose the origin, so Green's theorem applies on its disk and gives $\iint 0\,dA = 0$	No hole inside, no circulation.

$\oint = 2\pi$ **around the unit circle,** 0 **around the circle centered at** $(3, 0)$**, and** 2π **around the ellipse**	Green's theorem on the ring between the ellipse and the unit circle (where F is smooth) says the two loops around the origin give the same value. F is the gradient of the polar angle θ, which increases by 2π once around the origin, so no single-valued potential exists on the punctured plane (Mistake 4).

Problem 10

> Gradient descent on $L(w_1, w_2) = \frac{1}{2}(w_1^2 + 4w_2^2)$ moves along the field $V = -\nabla L$. A two-player game where one player minimizes and the other maximizes $w_1 w_2$ moves along $G = (-w_2, w_1)$. Compute $\int V \cdot dr$ along the segment from $(2, 1)$ to $(0, 0)$ and compare it with L. Then compute the circulation of V and of G counterclockwise around the circle of radius 2 about the origin, and say what each means for the dynamics.

$V = (-w_1, -4w_2)$, and $r(t) = (2 - 2t, 1 - t)$, $r' = (-2, -1)$	The straight segment, $0 \le t \le 1$.
$V \cdot r' = 2(2 - 2t) + 4(1 - t) = 8(1 - t)$, and $\int_0^1 8(1 - t)\, dt = 4$	Equal to $L(2, 1) - L(0, 0) = \frac{1}{2}(4 + 4) = 4$: the work of $-\nabla L$ is the drop in loss, along any path.
Circulation of V: $Q_x - P_y = 0 - 0 = 0$, so $\oint V \cdot dr = 0$	V is a gradient on the whole plane.
Circulation of G: $Q_x - P_y = 1 - (-1) = 2$, so $\oint G \cdot dr = 2 \cdot \pi \cdot 2^2 = 8\pi$	Green's theorem; directly, $G \cdot r' = 4\sin^2 t + 4\cos^2 t = 4$ around the circle, and $4 \cdot 2\pi = 8\pi$.
$\int V \cdot dr = 4 = L(2, 1) - L(0, 0)$; **circulation** 0 **for** V **and** $8\pi \approx 25.13$ **for** G	A gradient field cannot push an iterate around a loop, because each step lowers L. The game field has nonzero circulation, so it is not the gradient of any loss, and its updates rotate around the equilibrium at the origin instead of settling into it.

Where this goes wrong

1. Dropping |r′(t)| from ds

The arc-length integral is written as if ds were dt.

$r(t) = (3\cos t, 3\sin t)$, $0 \le t \le \pi$, and the density is $y = 3\sin t$	Right so far: Problem 1.
"$\int_C y\,ds = \int_0^{\pi} 3\sin t\,dt$."	The step that causes the mistake: $ds = \lVert r'(t)\rVert\,dt = 3\,dt$, because t is an angle, not a length.
Mass $= 6$ ✗	A third of the true mass. The answer changes with the parametrization: $r(t) = (3\cos 2t, 3\sin 2t)$ on $[0, \frac{\pi}{2}]$ would give 3. With the speed included it is 18 either way.

2. Going around the boundary clockwise

The ellipse is parametrized in the direction that first comes to mind, from the top.

Area $= \frac{1}{2}\oint_C x\,dy - y\,dx$ for the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$	Right so far: Problem 7.
"$x = 3\sin t$, $y = 2\cos t$."	This starts at $(0, 2)$ and moves to the right, so it runs clockwise, with the region on the right.
$x\,dy - y\,dx = -6\,dt$, so the area is -6π ✗	An area cannot be negative. Green's theorem needs the counterclockwise direction; reversing it flips the sign of every line integral. The area is 6π.

3. P_y − Q_x instead of Q_x − P_y

The two partial derivatives of Green's theorem are subtracted in the order they are written down.

$P = x - y^3$, $Q = x^3 + y$, $P_y = -3y^2$, $Q_x = 3x^2$	Right so far: Problem 6.

“The integrand is $P_y - Q_x = -3y^2 - 3x^2$.”	The order is $Q_x - P_y$: the x-derivative of the dy coefficient first.
$\oint_C = -\frac{3\pi}{2}$ ✗	The direct computation around the circle gives $\int_0^{2\pi} (\sin^4 t + \cos^4 t)\, dt = \frac{3\pi}{2}$, positive. A test that catches the order: for $F = (-y, x)$, which turns counterclockwise, $Q_x - P_y = 2 > 0$.

4. Zero curl read as conservative on a region with a hole

The cross-partials test is applied without checking where the field is defined.

$Q_x = P_y = \frac{y^2 - x^2}{(x^2 + y^2)^2}$ for the field of Problem 9	Right so far.
“$Q_x - P_y = 0$, so by Green's theorem every closed integral is 0.”	Green's theorem needs P and Q smooth on the whole region inside the curve, and the unit disk contains the origin, where F is undefined.
$\oint F \cdot dr = 0$ around the unit circle ✗	The direct computation gives $F \cdot r' = 1$ at every point and $\oint = 2\pi \approx 6.283$. The test proves a field conservative only on a region with no holes; around the origin, F is the gradient of the angle, which does not come back to its starting value.

5. dx = dt, and dy left as dt too

After choosing $x = t$, both differentials are replaced by dt.

$r(t) = (t, t^2)$, $P = t^2$, $Q = t^3$	Right so far: Problem 2.
“$\int_C P\, dx + Q\, dy = \int_0^1 (t^2 + t^3)\, dt$.”	$dx = dt$ is right, but $dy = 2t\, dt$ because $y = t^2$.
$W = \frac{1}{3} + \frac{1}{4} = \frac{7}{12} \approx 0.5833$ ✗	The integral of $F \cdot r'(t)$ needs the whole of $r'(t) = (1, 2t)$. With $dy = 2t\, dt$ the integrand is $t^2 + 2t^4$ and $W = \frac{11}{15} \approx 0.7333$.

More at eulerpad.app. This is one of 82 free practice sets with worked solutions at eulerpad.app/practice, alongside guides to the topics behind them.

EulerPad, the app, reads your handwritten math on iPad, iPhone and Android and checks it line by line. It's free to start.

eulerpad.app/practice/line-integrals-and-greens-theorem