

L'Hôpital's rule and indeterminate forms

Ten limits by L'Hôpital's rule: $0/0$ and ∞/∞ , a rule applied twice, $x \ln x$ and the $0 \log 0 = 0$ of entropy, x^x and $(1 + 2/x)^x$ by taking logs, a difference of fractions, and a limit the rule cannot find, with worked solutions and the mistakes of using it on a limit that is not indeterminate and of reading 1^∞ as 1.

BEFORE YOU START

When substitution gives $\frac{0}{0}$ or $\frac{\infty}{\infty}$, L'Hôpital's rule replaces the limit of a ratio with the limit of the ratio of the derivatives. It is how the convention $0 \log 0 = 0$ in entropy is justified, and why $x^2 e^{-x}$ and $\frac{\ln x}{\sqrt{x}}$ go to 0: exponentials beat powers, and powers beat logs. The work goes wrong in a few places: the rule used on a limit that is not indeterminate, the quotient rule used instead of differentiating top and bottom separately, 1^∞ read as 1, a log taken and never undone, and a conclusion drawn when the rule does not apply. These ten problems cover each of them.

- **L'Hôpital's rule.** Suppose f and g are differentiable near a (except possibly at a), $g'(x) \neq 0$ near a , and $\frac{f(x)}{g(x)}$ has the form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ as $x \rightarrow a$. If $\lim \frac{f'(x)}{g'(x)}$ exists (or is $\pm\infty$), then $\lim \frac{f(x)}{g(x)}$ equals it. The same holds for one-sided limits and for $a = \pm\infty$.
- f' and g' are differentiated **separately**. It is not the quotient rule.
- Check the form before every use. If substitution gives a number, that number is the limit and the rule does not apply.
- The other indeterminate forms are rewritten first. $0 \cdot \infty$: write the product as a quotient, $fg = \frac{f}{1/g}$. $\infty - \infty$: combine into one fraction. 1^∞ , 0^0 and ∞^0 : take the log, find the limit L of the log, and the answer is e^L .
- If $\lim \frac{f'}{g'}$ does not exist, the rule says nothing: the original limit may still exist (Problem 10).
- Derivatives used: $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$, $(e^x)' = e^x$, $(\ln x)' = \frac{1}{x}$, $(x^n)' = nx^{n-1}$.

Builds on: [Evaluating limits: factoring, conjugates and limits at infinity](#), [Differentiation rules: chain, product, quotient](#)

Problems

1. Find $\lim_{x \rightarrow 0} \frac{\sin 3x}{x}$.

difficulty ·

2. Find $\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2}$.

difficulty ·

3. Find $\lim_{x \rightarrow \infty} \frac{x^2}{e^x}$.

difficulty ··

4. Find $\lim_{x \rightarrow 0^+} x \ln x$.

difficulty ··

5. Find $\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}}$.

difficulty ··

6. Find $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$.

difficulty ··

7. Find $\lim_{x \rightarrow 0^+} x^x$.

difficulty ···

8. Find $\lim_{x \rightarrow \infty} \left(1 + \frac{2}{x}\right)^x$.

difficulty ···

9. Find $\lim_{x \rightarrow 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$.

difficulty ···

10. Find $\lim_{x \rightarrow \infty} \frac{x + \sin x}{x}$. What happens if you use L'Hôpital's rule?

difficulty ···

Answers

1. 3

2. $\frac{1}{2}$

3. 0

4. 0

5. 0

6. $\frac{1}{2}$

7. 1

8. $e^2 \approx 7.389$

9. 0

10. 1; L'Hôpital's rule does not apply, because $\lim(1 + \cos x)$ does not exist

Worked solutions

Problem 1

Find $\lim_{x \rightarrow 0} \frac{\sin 3x}{x}$.

At $x = 0$: $\frac{\sin 0}{0} = \frac{0}{0}$.

Indeterminate, so the rule may apply.

$\lim_{x \rightarrow 0} \frac{3 \cos 3x}{1}$

Differentiate top and bottom separately; the chain rule gives the 3.

3

$3 \cos 0 = 3$. Near 0, $\sin 3x \approx 3x$.

Problem 2

Find $\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2}$.

At $x = 0$: $\frac{1-1-0}{0} = \frac{0}{0}$.

Indeterminate.

$\lim_{x \rightarrow 0} \frac{e^x - 1}{2x}$, again $\frac{0}{0}$

One application. Check the form again before the next one.

$\lim_{x \rightarrow 0} \frac{e^x}{2}$

A second application.

$\frac{1}{2}$

$\frac{e^0}{2}$. It is the $\frac{x^2}{2}$ term of the Taylor series $e^x = 1 + x + \frac{x^2}{2} + \dots$.

Problem 3

Find $\lim_{x \rightarrow \infty} \frac{x^2}{e^x}$.

As $x \rightarrow \infty$: $\frac{\infty}{\infty}$.

Indeterminate.

$\lim_{x \rightarrow \infty} \frac{2x}{e^x}$, still $\frac{\infty}{\infty}$

The power drops by one; the exponential does not change.

$\lim_{x \rightarrow \infty} \frac{2}{e^x}$	A second application. The top is now a constant.
0	$e^x \rightarrow \infty$. The same argument, applied n times, shows $x^n e^{-x} \rightarrow 0$ for every n .

Problem 4

Find $\lim_{x \rightarrow 0^+} x \ln x$.	
As $x \rightarrow 0^+$: $0 \cdot (-\infty)$.	Indeterminate, but not a quotient yet.
$x \ln x = \frac{\ln x}{1/x}$, which has the form $\frac{-\infty}{\infty}$.	Move x to the denominator as $\frac{1}{x}$. Keeping $\ln x$ on top makes the derivatives simple.
$\lim_{x \rightarrow 0^+} \frac{1/x}{-1/x^2} = \lim_{x \rightarrow 0^+} (-x)$	Differentiate top and bottom; $\frac{1/x}{-1/x^2} = -x$.
0	This is why entropy uses $0 \log 0 = 0$: $p \log p \rightarrow 0$ as $p \rightarrow 0^+$, so an outcome of probability 0 adds nothing.

Problem 5

Find $\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}}$.	
As $x \rightarrow \infty$: $\frac{\infty}{\infty}$.	Indeterminate.
$\lim_{x \rightarrow \infty} \frac{1/x}{1/(2\sqrt{x})} = \lim_{x \rightarrow \infty} \frac{2\sqrt{x}}{x}$	$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$.
$= \lim_{x \rightarrow \infty} \frac{2}{\sqrt{x}}$	$\frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$.
0	Any positive power of x eventually outgrows $\ln x$, even $x^{1/2}$.

Problem 6

Find $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$.

At $x = 0$: $\frac{1-1}{0} = \frac{0}{0}$.

Indeterminate.

$\lim_{x \rightarrow 0} \frac{\sin x}{2x}$, still $\frac{0}{0}$

$(1 - \cos x)' = \sin x$.

$\lim_{x \rightarrow 0} \frac{\cos x}{2}$

A second application.

$\frac{1}{2}$

$\frac{\cos 0}{2}$. So $\cos x \approx 1 - \frac{x^2}{2}$ near 0.

Problem 7

Find $\lim_{x \rightarrow 0^+} x^x$.

As $x \rightarrow 0^+$: 0^0 .

Indeterminate: a base going to 0 pulls toward 0, an exponent going to 0 pulls toward 1.

Let $y = x^x$; then $\ln y = x \ln x$.

Take the log to bring the exponent down.

$\lim_{x \rightarrow 0^+} \ln y = 0$

Problem 4.

1

$y = e^{\ln y}$, and e^x is continuous, so $y \rightarrow e^0 = 1$. The answer is not 0, which is the limit of the log (mistake 4).

Problem 8

Find $\lim_{x \rightarrow \infty} \left(1 + \frac{2}{x}\right)^x$.

As $x \rightarrow \infty$: 1^∞ .

Indeterminate: the base goes to 1, but the exponent grows without bound.

$\ln y = x \ln \left(1 + \frac{2}{x}\right) = \frac{\ln(1 + 2/x)}{1/x}$, which has the form $\frac{0}{0}$.	Take the log, then write the product as a quotient.
$\lim_{x \rightarrow \infty} \frac{\frac{1}{1 + 2/x} \cdot \left(-\frac{2}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x \rightarrow \infty} \frac{2}{1 + 2/x} = 2$	The chain rule on the top: $\left(\frac{2}{x}\right)' = -\frac{2}{x^2}$. The $-\frac{1}{x^2}$ cancels.
$e^2 \approx 7.389$	$y = e^{\ln y} \rightarrow e^2$. This is compound interest at a rate of 200% compounded ever more often.

Problem 9

Find $\lim_{x \rightarrow 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$.	
As $x \rightarrow 0^+$: $\infty - \infty$.	Indeterminate; from the left it is $-\infty + \infty$.
$\frac{1}{x} - \frac{1}{\sin x} = \frac{\sin x - x}{x \sin x}$, which has the form $\frac{0}{0}$.	Combine into one fraction over the common denominator.
$\lim_{x \rightarrow 0} \frac{\cos x - 1}{\sin x + x \cos x}$, still $\frac{0}{0}$	The product rule on the bottom: $(x \sin x)' = \sin x + x \cos x$.
$\lim_{x \rightarrow 0} \frac{-\sin x}{2 \cos x - x \sin x} = \frac{0}{2}$	Again: $(\sin x + x \cos x)' = \cos x + \cos x - x \sin x$.
0	From both sides. Near 0, $\sin x \approx x - \frac{x^3}{6}$, so $\frac{1}{\sin x} \approx \frac{1}{x} \left(1 + \frac{x^2}{6}\right)$ and the difference is about $-\frac{x}{6}$.

Problem 10

Find $\lim_{x \rightarrow \infty} \frac{x + \sin x}{x}$. What happens if you use L'Hôpital's rule?

As $x \rightarrow \infty$: $\frac{\infty}{\infty}$.

Indeterminate.

The rule gives $\lim_{x \rightarrow \infty} \frac{1 + \cos x}{1}$, which does not exist.

$1 + \cos x$ keeps swinging between 0 and 2. The rule needs this limit to exist, so it gives no answer here, and it does not show that the original limit fails to exist.

$$\frac{x + \sin x}{x} = 1 + \frac{\sin x}{x}$$

Divide top and bottom by x instead.

$$-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}, \text{ so } \frac{\sin x}{x} \rightarrow 0.$$

$|\sin x| \leq 1$, and both bounds go to 0 (squeeze).

1; L'Hôpital's rule does not apply, because $\lim(1 + \cos x)$ does not exist

The rule is a sufficient condition, not a test: when the ratio of derivatives has no limit, use another method.

Where this goes wrong

1. Using the rule on a limit that is not indeterminate

The rule is used so often that it becomes the first step for every fraction.

$$\lim_{x \rightarrow 1} \frac{x^2 + 1}{x + 1}$$

A limit to find.

“It is a fraction, so differentiate top and bottom.”

The habit that causes the mistake: skipping the check of the form.

$$\lim_{x \rightarrow 1} \frac{2x}{1} = 2 \quad \text{✗}$$

At $x = 1$ the fraction is $\frac{2}{2} = 1$, not $\frac{0}{0}$ or $\frac{\infty}{\infty}$, so the rule does not apply, and the limit is 1 by substitution. Check the form every time, including before each repeat application.

2. The quotient rule instead of two separate derivatives

L'Hôpital's rule involves a fraction and derivatives, and the quotient rule is the derivative of a fraction.

$$\lim_{x \rightarrow 0} \frac{\sin 3x}{x}, \text{ form } \frac{0}{0}$$

Right so far: Problem 1, step 1.

“Differentiate the fraction.”

The habit that causes the mistake: the rule remembered as “take the derivative”.

$$\lim_{x \rightarrow 0} \frac{3x \cos 3x - \sin 3x}{x^2} = 0 \text{ ✗}$$

That is the limit of the derivative of $\frac{\sin 3x}{x}$, the slope of its graph near 0, which is a different question. The rule differentiates the top and the bottom separately: $\frac{3 \cos 3x}{1} \rightarrow 3$.

3. Reading 1^∞ as 1

One to any power is one, so a base that goes to 1 seems to settle the limit.

$$\lim_{x \rightarrow \infty} \left(1 + \frac{2}{x}\right)^x$$

Right so far: Problem 8.

“The base goes to 1, and $1^x = 1$.”

The habit that causes the mistake: letting the base reach its limit while the exponent is still finite.

$$\lim_{x \rightarrow \infty} \left(1 + \frac{2}{x}\right)^x = 1 \text{ ✗}$$

The base is never exactly 1, and the exponent grows as fast as the base approaches 1. At $x = 100$ the value is already $1.02^{100} \approx 7.24$. Take the log: $x \ln \left(1 + \frac{2}{x}\right) \rightarrow 2$, so the limit is $e^2 \approx 7.389$.

4. Forgetting to undo the log

After taking a log, the limit that comes out looks like the end of the problem.

$$y = x^x, \text{ so } \ln y = x \ln x \rightarrow 0 \text{ as } x \rightarrow 0^+$$

Right so far: Problem 7, steps 2 and 3.

“The limit is 0.”

The habit that causes the mistake:
reporting the limit of $\ln y$ as the limit of
 y .

$$\lim_{x \rightarrow 0^+} x^x = 0 \text{ ✗}$$

0 is the limit of $\ln y$. The limit of y is
 $e^0 = 1$. A check: $0.01^{0.01} \approx 0.955$, near 1
and nowhere near 0.

5. Concluding the limit does not exist

When the rule produces something with no limit, it looks like the answer is "no limit".

$$\lim_{x \rightarrow \infty} \frac{x + \sin x}{x}, \text{ form } \frac{\infty}{\infty}$$

Right so far: Problem 10.

“L'Hôpital: $\lim(1 + \cos x)$ does not exist.”

Right so far as a fact about $1 + \cos x$.

$$\text{So } \lim_{x \rightarrow \infty} \frac{x + \sin x}{x} \text{ does not exist ✗}$$

The rule only says that if the ratio of
derivatives has a limit, the original ratio
has the same one. It says nothing when
the ratio of derivatives has no limit.
Here $\frac{x + \sin x}{x} = 1 + \frac{\sin x}{x} \rightarrow 1$.

eulerpad.app/practice/lhopitals-rule