Laplace transforms: the table, the shift rules and initial-value problems

Ten problems on Laplace transforms: e^(3t) and t from the definition with the region of convergence, a sum of a constant, a power and a sine by linearity and the table, the first shift theorem for e^(−2t) cos 3t and t³e^(4t), an inverse transform by partial fractions, an inverse by completing the square into a damped oscillation, a first-order initial-value problem, a second-order initial-value problem with nonzero initial conditions, the transform of a ramp switched on at t = 2, a spring switched on by a unit step at t = π, and the impulse response, step response and final value of y″ + 2y′ + 5y from the transfer function and the convolution theorem, with worked solutions and the mistakes of dropping the initial conditions from the derivative rule, transforming a product as the product of the transforms, shifting the wrong way, transforming u(t − c)f(t) as e^(−cs)F(s) and completing the square in the denominator only.

BEFORE YOU START

The Laplace transform turns a function of time into a function of a new variable s by an improper integral, and its value is that it turns calculus into algebra. Differentiating $y(t)$ becomes multiplying $Y(s)$ by s and subtracting the initial value, so a linear differential equation with constant coefficients becomes a linear equation for $Y(s)$, with the initial conditions already built in. Solve it for Y, split the answer into pieces that appear in a short table, and read the solution back off the table. Two shift rules handle exponential factors and forcing that switches on at a later time, and the convolution theorem says that the transform of a system's response is its transfer function times the transform of the input, which is how control engineers and signal processing describe linear systems. These ten problems build the table from the definition, then use it to solve initial-value problems, including ones with switched-on forcing and an impulse.

- **Definition.** $\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t)\,dt$, for the values of s where the integral converges. If $|f(t)| \le Me^{at}$, it converges for every $s > a$.
- **Table.** $\mathcal{L}\{1\} = \frac{1}{s}$, $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$, $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$, $\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}$, $\mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2}$.
- **Linearity.** $\mathcal{L}\{af + bg\} = aF + bG$. There is no such rule for a product: $\mathcal{L}\{fg\}$ is not FG.
- **First shift.** $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$: multiplying by e^{at} replaces s by $s - a$.
- **Derivatives.** $\mathcal{L}\{y'\} = sY - y(0)$ and $\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0)$.
- **Unit step and second shift.** $u(t-c)$ is 0 for $t < c$ and 1 for $t \ge c$. Then $\mathcal{L}\{u(t-c)f(t-c)\} = e^{-cs}F(s)$, and so $\mathcal{L}\{u(t-c)g(t)\} = e^{-cs}\mathcal{L}\{g(t+c)\}$.

- **Convolution and impulses.** $(f * g)(t) = \int_0^t f(\tau) g(t-\tau)\,d\tau$ has transform $F(s)G(s)$, and the unit impulse has $\mathcal{L}\{\delta(t)\} = 1$.
- **Final value.** If every pole of $sF(s)$ has negative real part, then $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$.

Builds on: Improper integrals: infinite limits, infinite integrands and comparison, Partial fractions and integrating rational functions, Second-order linear differential equations with constant coefficients

Problems

1. Compute $\mathcal{L}\{e^{3t}\}$ and $\mathcal{L}\{t\}$ from the definition, and say for which s each integral converges. difficulty ·

2. Find the transform of $f(t) = 4 - 3t^2 + 5\sin 2t$. difficulty ·

3. Use the first shift theorem to find $\mathcal{L}\{e^{-2t}\cos 3t\}$ and $\mathcal{L}\{t^3 e^{4t}\}$. difficulty ··

4. Find the inverse transform of difficulty ·

$$F(s) = \frac{5s + 1}{(s-1)(s+2)}.$$

5. Find the inverse transform of difficulty ··

$$F(s) = \frac{2s + 3}{s^2 + 4s + 13}.$$

6. Solve $y' + 3y = e^{-t}$ with $y(0) = 2$ using the Laplace transform. difficulty ··

7. Solve $y'' + 3y' + 2y = 4$ with $y(0) = 1$ and $y'(0) = 2$. difficulty ···

8. Find the transform of the ramp that switches on at $t = 2$: difficulty ··

$$g(t) = \begin{cases} 0 & t < 2 \\ t & t \geq 2. \end{cases}$$

9. A spring at rest satisfies $y'' + 4y = u(t - \pi)$, $y(0) = 0$, $y'(0) = 0$: a constant force switches on at $t = \pi$. Find $y(t)$ and $y(3\pi/2)$. difficulty ···

10. A system obeys $y'' + 2y' + 5y = f(t)$ and starts at rest. Find its transfer function $H(s)$, its impulse response $h(t)$ for $f = \delta(t)$, its step response for $f = 1$, and the value the step response settles to. difficulty ···

Answers

1. $\mathcal{L}\{e^{3t}\} = \frac{1}{s-3}$ for $s > 3$ and $\mathcal{L}\{t\} = \frac{1}{s^2}$ for $s > 0$
2. $F(s) = \frac{4}{s} - \frac{6}{s^3} + \frac{10}{s^2+4}$, for $s > 0$
3. $\mathcal{L}\{e^{-2t}\cos 3t\} = \frac{s+2}{s^2+4s+13}$ for $s > -2$, and $\mathcal{L}\{t^3 e^{4t}\} = \frac{6}{(s-4)^4}$ for $s > 4$
4. $f(t) = 2e^{t} + 3e^{-2t}$
5. $f(t) = 2e^{-2t}\cos 3t - \frac{1}{3}e^{-2t}\sin 3t$
6. $y(t) = \frac{3}{2}e^{-3t} + \frac{1}{2}e^{-t}$
7. $y(t) = 2 - e^{-2t}$
8. $G(s) = \frac{(1+2s)e^{-2s}}{s^2}$
9. $y(t) = \frac{1}{4}u(t-\pi)\big(1 - \cos 2(t-\pi)\big) = \frac{1}{4}u(t-\pi)(1-\cos 2t)$, so $y(3\pi/2) = \frac{1}{2}$
10. $h(t) = \frac{1}{2}e^{-t}\sin 2t$, and the step response is $\frac{1}{5} - \frac{1}{5}e^{-t}\cos 2t - \frac{1}{10}e^{-t}\sin 2t$, which settles to $\frac{1}{5}$

Worked solutions

Problem 1

Compute $\mathcal{L}\{e^{3t}\}$ and $\mathcal{L}\{t\}$ from the definition, and say for which s each integral converges.

$\int_0^\infty e^{-st}e^{3t}\,dt = \int_0^\infty e^{-(s-3)t}\,dt$	Combine the exponentials.
$= \lim_{b\to\infty} \dfrac{1-e^{-(s-3)b}}{s-3}$	Replace ∞ by b, integrate, then take the limit.
$\mathcal{L}\{e^{3t}\} = \dfrac{1}{s-3}$ for $s>3$	The exponential tends to 0 only when $s-3>0$; for $s\le 3$ the integral diverges.
$\int_0^\infty te^{-st}\,dt = \Big[-\dfrac{te^{-st}}{s}\Big]_0^\infty + \dfrac{1}{s}\int_0^\infty e^{-st}\,dt$	Integration by parts with $u=t$, $dv=e^{-st}dt$. The bracket is 0 for $s>0$.
$\mathcal{L}\{e^{3t}\} = \frac{1}{s-3}$ **for** $s>3$ **and** $\mathcal{L}\{t\} = \frac{1}{s^2}$ **for** $s>0$	At $s=4$ these are 1 and $\frac{1}{16}$. The transform of t is the transform of 1, which is $\frac{1}{s}$, divided by s once more.

Problem 2

Find the transform of $f(t) = 4 - 3t^2 + 5\sin 2t$.

$\mathcal{L}\{4\} = \dfrac{4}{s}$	Four times $\mathcal{L}\{1\}$.
$\mathcal{L}\{t^2\} = \dfrac{2!}{s^3} = \dfrac{2}{s^3}$	The table with $n=2$; do not forget the factorial.
$\mathcal{L}\{\sin 2t\} = \dfrac{2}{s^2+4}$	The table with $b=2$: b on top, b^2 below.
$F(s) = \dfrac{4}{s} - 3\cdot\dfrac{2}{s^3} + 5\cdot\dfrac{2}{s^2+4}$	Linearity: the constants come through unchanged.
$F(s) = \frac{4}{s} - \frac{6}{s^3} + \frac{10}{s^2+4}$, **for** $s>0$	At $s=1$ it is $4-6+2=0$, so $\int_0^\infty e^{-t}f(t)\,dt = 0$.

Problem 3

Use the first shift theorem to find $\mathcal{L}\{e^{-2t}\cos 3t\}$ and $\mathcal{L}\{t^3 e^{4t}\}$.

$\mathcal{L}\{\cos 3t\} = \dfrac{s}{s^2+9}$	The table with $b = 3$.
e^{-2t} means $a = -2$, so s becomes $s + 2$	$F(s-a)$ with $a = -2$ is $F(s+2)$.
$\mathcal{L}\{e^{-2t}\cos 3t\} = \dfrac{s+2}{(s+2)^2+9}$	Replace every s, in the numerator too.
$\mathcal{L}\{t^3\} = \dfrac{6}{s^4}$, and e^{4t} turns s into $s-4$	$3! = 6$.
$\mathcal{L}\{e^{-2t}\cos 3t\} = \frac{s+2}{s^2+4s+13}$ **for** $s > -2$, **and** $\mathcal{L}\{t^3 e^{4t}\} = \frac{6}{(s-4)^4}$ **for** $s > 4$	The shift moves the region of convergence too. At $s = 1$ the first is $\frac{3}{18} = \frac{1}{6}$; shifting the other way gives -0.1 (Mistake 3).

Problem 4

Find the inverse transform of

$$F(s) = \frac{5s+1}{(s-1)(s+2)}.$$

$\dfrac{5s+1}{(s-1)(s+2)} = \dfrac{A}{s-1} + \dfrac{B}{s+2}$	Two distinct linear factors.
$A = \dfrac{5+1}{1+2} = 2$	Cover up $s-1$ and set $s = 1$.
$B = \dfrac{-10+1}{-2-1} = 3$	Cover up $s+2$ and set $s = -2$.
$2(s+2) + 3(s-1) = 5s+1$	Check by recombining.
$f(t) = 2e^{t} + 3e^{-2t}$	Each $\frac{1}{s-a}$ is e^{at}. Check: $f(0) = 5$, and $\lim_{s\to\infty} sF(s) = 5$ as well.

Problem 5

Find the inverse transform of

$$F(s) = \frac{2s + 3}{s^2 + 4s + 13}.$$

$s^2 + 4s + 13 = (s + 2)^2 + 9$	Complete the square; the roots $-2 \pm 3i$ are complex, so there are no real linear factors.
$2s + 3 = 2(s + 2) - 1$	Rewrite the numerator in terms of $s + 2$ as well.
$F(s) = \dfrac{2(s + 2)}{(s + 2)^2 + 9} - \dfrac{1}{3} \cdot \dfrac{3}{(s + 2)^2 + 9}$	A shifted cosine and a shifted sine, with the 3 the sine needs on top.
$\dfrac{s + 2}{(s + 2)^2 + 9} \to e^{-2t}\cos 3t,$ $\dfrac{3}{(s + 2)^2 + 9} \to e^{-2t}\sin 3t$	The first shift theorem with $a = -2$, read backward.
$f(t) = 2e^{-2t}\cos 3t - \frac{1}{3}e^{-2t}\sin 3t$	A damped oscillation: decay rate 2 from the real part of the roots, frequency 3 from the imaginary part. Check: $f(0) = 2$, and $f'(0) = -4 - 1 = -5$.

Problem 6

Solve $y' + 3y = e^{-t}$ with $y(0) = 2$ using the Laplace transform.

$sY - 2 + 3Y = \dfrac{1}{s + 1}$	Transform both sides; $\mathcal{L}\{y'\} = sY - y(0)$.
$Y = \dfrac{2}{s + 3} + \dfrac{1}{(s + 1)(s + 3)}$	Collect $(s + 3)Y$ and divide.
$\dfrac{1}{(s + 1)(s + 3)} = \dfrac{1/2}{s + 1} - \dfrac{1/2}{s + 3}$	Cover-up: $\frac{1}{2}$ at $s = -1$ and $-\frac{1}{2}$ at $s = -3$.
$Y = \dfrac{3/2}{s + 3} + \dfrac{1/2}{s + 1}$	Combine the two terms in $\frac{1}{s+3}$.

$y(t) = \frac{3}{2}e^{-3t} + \frac{1}{2}e^{-t}$	Check: $y(0) = 2$, and $y' + 3y = -\frac{9}{2}e^{-3t} - \frac{1}{2}e^{-t} + \frac{9}{2}e^{-3t} + \frac{3}{2}e^{-t} = e^{-t}$.

Problem 7

Solve $y'' + 3y' + 2y = 4$ with $y(0) = 1$ and $y'(0) = 2$.

$s^2Y - s - 2 + 3(sY - 1) + 2Y = \dfrac{4}{s}$	$\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0)$ and $\mathcal{L}\{y'\} = sY - y(0)$.
$(s^2 + 3s + 2)Y = s + 5 + \dfrac{4}{s}$	Move the initial-condition terms $s + 2 + 3$ to the right.
$Y = \dfrac{s^2 + 5s + 4}{s(s+1)(s+2)}$	Put the right side over s and divide by $(s+1)(s+2)$.
$Y = \dfrac{(s+1)(s+4)}{s(s+1)(s+2)} = \dfrac{s+4}{s(s+2)}$	The factor $s + 1$ cancels, so the e^{-t} mode never appears for these initial values.
$\dfrac{s+4}{s(s+2)} = \dfrac{2}{s} - \dfrac{1}{s+2}$	Cover-up: 2 at $s = 0$ and $\frac{2}{-2} = -1$ at $s = -2$.
$y(t) = 2 - e^{-2t}$	Check: $y(0) = 1$, $y'(0) = 2$, and $y'' + 3y' + 2y = -4e^{-2t} + 6e^{-2t} + 4 - 2e^{-2t} = 4$. Leaving out the initial conditions gives a solution with $y(0) = 0$ (Mistake 1).

Problem 8

Find the transform of the ramp that switches on at $t = 2$:

$$g(t) = \begin{cases} 0 & t < 2 \\ t & t \geq 2. \end{cases}$$

$g(t) = u(t-2)\,t$	The unit step switches t on at $t = 2$.
$\mathcal{L}\{u(t-2)\,t\} = e^{-2s}\mathcal{L}\{t+2\}$	The second shift theorem needs the function written in terms of $t - 2$; equivalently, shift its argument by $+2$.

$\mathcal{L}\{t+2\} = \dfrac{1}{s^2} + \dfrac{2}{s}$	$t+2$ is the same ramp, measured from the switch-on time: it starts at the value 2.
$G(s) = e^{-2s}\left(\dfrac{1}{s^2} + \dfrac{2}{s}\right)$	Multiply by e^{-2s}.
$G(s) = \frac{(1+2s)e^{-2s}}{s^2}$	Check from the definition: $\int_2^\infty te^{-st}\,dt$ at $s=1$ is $3e^{-2} \approx 0.406$, and $G(1) = 3e^{-2}$. Writing e^{-2s}/s^2 drops the jump to 2 (Mistake 4).

Problem 9

> A spring at rest satisfies $y'' + 4y = u(t-\pi)$, $y(0)=0$, $y'(0)=0$: a constant force switches on at $t=\pi$. Find $y(t)$ and $y(3\pi/2)$.

$(s^2+4)Y = \dfrac{e^{-\pi s}}{s}$	$\mathcal{L}\{u(t-\pi)\} = e^{-\pi s}/s$, and both initial values are 0.
$Y = e^{-\pi s}\cdot\dfrac{1}{s(s^2+4)}$	Keep the exponential aside; it only shifts the answer.
$\dfrac{1}{s(s^2+4)} = \dfrac{1}{4}\left(\dfrac{1}{s} - \dfrac{s}{s^2+4}\right)$	Partial fractions: $\frac{1}{4}$ at $s=0$, and the rest is $\frac{Bs+C}{s^2+4}$ with $B=-\frac{1}{4}$, $C=0$.
$\dfrac{1}{s(s^2+4)} \to \frac{1}{4}(1-\cos 2t)$	The table, without the shift.
$y(t) = \frac{1}{4}u(t-\pi)\big(1-\cos 2(t-\pi)\big) = \frac{1}{4}u(t-\pi)(1-\cos 2t)$, **so** $y(3\pi/2) = \frac{1}{2}$	The spring stays at rest until $t=\pi$, then oscillates between 0 and $\frac{1}{2}$ about its new equilibrium $\frac{1}{4}$. Here $\cos 2(t-\pi) = \cos 2t$ because 2π is a full period; in general the shift must stay inside.

Problem 10

> A system obeys $y'' + 2y' + 5y = f(t)$ and starts at rest. Find its transfer function $H(s)$, its impulse response $h(t)$ for $f = \delta(t)$, its step response for $f = 1$, and the value the step response settles to.

$(s^2 + 2s + 5)Y = F(s)$, so $H(s) = \dfrac{Y}{F} = \dfrac{1}{s^2 + 2s + 5}$	At rest, the initial-condition terms are all 0.
$h(t) = \mathcal{L}^{-1}\left\{ \dfrac{1}{(s+1)^2 + 4} \right\} = \frac{1}{2} e^{-t} \sin 2t$	$\mathcal{L}\{\delta\} = 1$, so $Y = H$. Complete the square and put the needed 2 on top: $\frac{1}{2} \cdot \frac{2}{(s+1)^2+4}$.
Step: $Y = \dfrac{H(s)}{s}$, which is the transform of $\int_0^t h(\tau)\, d\tau$	The convolution theorem: the response is $h * f$, and with $f = 1$ the convolution is the running integral of h.
$\dfrac{1}{s(s^2 + 2s + 5)} = \dfrac{1/5}{s} - \dfrac{1}{5} \cdot \dfrac{s + 2}{s^2 + 2s + 5}$	$A = \frac{1}{5}$ at $s = 0$; then $B = -\frac{1}{5}$ from the s^2 terms and $C = -\frac{2}{5}$ from the s terms.
$\dfrac{s+2}{(s+1)^2+4} = \dfrac{s+1}{(s+1)^2+4} + \dfrac{1}{2} \cdot \dfrac{2}{(s+1)^2+4}$	Shift the numerator as well as the denominator.
$h(t) = \frac{1}{2} e^{-t} \sin 2t$, **and the step response is** $\frac{1}{5} - \frac{1}{5} e^{-t} \cos 2t - \frac{1}{10} e^{-t} \sin 2t$, **which settles to** $\frac{1}{5}$	The final-value theorem agrees: $\lim_{s \to 0} s \cdot \frac{H(s)}{s} = H(0) = \frac{1}{5}$, allowed because the poles $-1 \pm 2i$ are in the left half-plane. The response starts at 0 with slope 0, as a system at rest must.

Where this goes wrong

1. Dropping the initial conditions from L{y''}

The derivative rule is remembered as "differentiation is multiplication by s", without the terms that carry the starting values.

$y'' + 3y' + 2y = 4$, $y(0) = 1$, $y'(0) = 2$	Right so far: Problem 7.

"$\mathcal{L}\{y''\} = s^2Y$ and $\mathcal{L}\{y'\} = sY$."	Those are the rules only when $y(0) = y'(0) = 0$. In general $\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0)$ and $\mathcal{L}\{y'\} = sY - y(0)$, and that is where the initial conditions enter.
$Y = \dfrac{4}{s(s+1)(s+2)}$, so $y = 2 - 4e^{-t} + 2e^{-2t}$ ✗	This solves the equation, but with $y(0) = 0$ and $y'(0) = 0$; it is the right answer to a different problem. The correct solution is $y = 2 - e^{-2t}$. Checking $y(0)$ at the end catches the slip at once.

2. The transform of a product taken as the product of the transforms

Linearity lets sums be transformed term by term, and products look as if they should work the same way.

$\mathcal{L}\{t\} = \frac{1}{s^2}$ and $\mathcal{L}\{e^{2t}\} = \frac{1}{s-2}$	Right so far: the table.
"$\mathcal{L}\{te^{2t}\} = \mathcal{L}\{t\}\,\mathcal{L}\{e^{2t}\}$."	There is no product rule. A factor e^{at} is handled by the first shift theorem: $\mathcal{L}\{te^{2t}\} = \frac{1}{(s-2)^2}$.
$\mathcal{L}\{te^{2t}\} = \dfrac{1}{s^2(s-2)}$ ✗	At $s = 3$ this gives $\frac{1}{9} \approx 0.111$ instead of 1. The product $\frac{1}{s^2} \cdot \frac{1}{s-2}$ is the transform of the convolution $t * e^{2t} = \frac{1}{4}(e^{2t} - 1 - 2t)$, not of the product te^{2t}.

3. Shifting the wrong way

The factor e^{-2t} has a minus sign, and the minus sign is copied into s.

$\mathcal{L}\{\cos 3t\} = \dfrac{s}{s^2+9}$	Right so far: Problem 3.
"e^{-2t} shifts s to $s - 2$."	The rule is $e^{at}f(t) \to F(s-a)$, and here $a = -2$, so s becomes $s - (-2) = s + 2$. A decaying factor moves the transform to the left, where s is smaller.

$\mathcal{L}\{e^{-2t}\cos 3t\} = \dfrac{s-2}{(s-2)^2+9}$ ✗	That is the transform of the growing $e^{2t}\cos 3t$. At $s=1$ it gives $-\frac{1}{10}$, while the right value $\frac{s+2}{(s+2)^2+9}$ gives $\frac{1}{6}$, which is what $\int_0^\infty e^{-t}e^{-2t}\cos 3t\,dt$ comes to. The wrong formula even has the wrong region of convergence, $s>2$ instead of $s>-2$.

4. u(t – c)f(t) transformed as e^(−cs)F(s)

The second shift theorem is remembered as "a step at c multiplies by e^{-cs}", without the shift of f.

$g(t) = u(t-2)\,t$	Right so far: Problem 8.
"$\mathcal{L}\{u(t-2)\,t\} = e^{-2s}\mathcal{L}\{t\}$."	The theorem is $\mathcal{L}\{u(t-c)f(t-c)\} = e^{-cs}F(s)$: the function must be written in terms of $t-c$ first, and $t = (t-2)+2$.
$G(s) = \dfrac{e^{-2s}}{s^2}$ ✗	This is the transform of $u(t-2)(t-2)$, a ramp that starts from 0 at $t=2$. At $s=1$ it gives $e^{-2}\approx 0.135$ instead of $3e^{-2}\approx 0.406$; the missing $\frac{2e^{-2s}}{s}$ is the jump of height 2.

5. Completing the square in the denominator only

The denominator is rewritten as $(s+2)^2+9$, but the numerator $2s+3$ is read as $2s$ plus a constant.

$F(s) = \dfrac{2s+3}{(s+2)^2+9}$	Right so far: Problem 5.
"$\dfrac{2s}{(s+2)^2+9} \to 2e^{-2t}\cos 3t$ and $\dfrac{3}{(s+2)^2+9} \to e^{-2t}\sin 3t$."	The shifted cosine has $s+2$ on top, not s. The numerator must be written as $2(s+2)-1$, which leaves -1, not 3, for the sine.
$f(t) = 2e^{-2t}\cos 3t + e^{-2t}\sin 3t$ ✗	Its transform is $\frac{2s+7}{s^2+4s+13}$, not $\frac{2s+3}{s^2+4s+13}$. Its slope at 0 is -1, while the right answer $2e^{-2t}\cos 3t - \frac{1}{3}e^{-2t}\sin 3t$ has slope -5.

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