Derivatives of inverse functions and inverse trig functions

Ten problems on the derivatives of inverse functions: the slope of an inverse at a point without a formula for the inverse, the derivatives of arccos and arctan from implicit differentiation, the chain rule through arctan and arcsin, an antiderivative of arcsin checked by differentiating, the derivative of the logit as the inverse of the sigmoid, the second derivative of an inverse, a branch that sets the sign, a camera tracking a rocket and the best place to stand in front of a painting, with worked solutions and the mistakes of reading f^{-1} as $1/f$, evaluating f' at the wrong point, giving arccos a positive slope, dropping the inner derivative and choosing the wrong branch.

BEFORE YOU START

An inverse function undoes a function: if $y = f(x)$ then $x = f^{-1}(y)$. Its graph is the graph of f reflected in the line $y = x$, and reflecting a line in $y = x$ turns slope m into slope $1/m$. That is the whole rule: the slope of f^{-1} at y is one over the slope of f at the matching point $x = f^{-1}(y)$. It gives the slope of an inverse that has no formula, the derivatives of $\arcsin$, $\arccos$ and $\arctan$, and the derivative of the logit, the inverse of the sigmoid. These ten problems use the rule at a point, derive the inverse trig derivatives and chain them, then put them to work on an antiderivative, a second derivative, a branch, a related-rates problem and an optimization. The five mistakes at the end are the ones that give a believable number: f^{-1} read as $1/f$, f' evaluated at y instead of at x, $\arccos$ given a positive slope, the inner derivative dropped, and the wrong branch of a square root.

- **Inverse.** If f is one-to-one on an interval, f^{-1} is defined by $f^{-1}(y) = x$ exactly when $f(x) = y$. Then $f(f^{-1}(y)) = y$ and $f^{-1}(f(x)) = x$. The notation f^{-1} means the inverse function, not $1/f$.
- **The inverse function rule.** If f is differentiable at $x = f^{-1}(y)$ and $f'(x) \neq 0$, then $(f^{-1})'(y) = \dfrac{1}{f'(f^{-1}(y))}$. In Leibniz notation, $\dfrac{dx}{dy} = 1 \Big/ \dfrac{dy}{dx}$, with both derivatives taken at the same point (x, y) of the graph.
- **Principal branches.** $\arcsin : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\arccos : [-1, 1] \to [0, \pi]$ and $\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ are the inverses of $\sin$, $\cos$ and $\tan$ restricted to those ranges.
- **Derivatives.** $(\arcsin x)' = \dfrac{1}{\sqrt{1 - x^2}}$ and $(\arccos x)' = -\dfrac{1}{\sqrt{1 - x^2}}$ for $-1 < x < 1$, and $(\arctan x)' = \dfrac{1}{1 + x^2}$ for every x. The implicit-differentiation page derives the first; Problem 3 derives the other two.
- $\sigma(t) = 1/(1 + e^{-t})$ is the sigmoid, with $\sigma' = \sigma(1 - \sigma)$, and $\ln$ is the natural log.

Builds on: Differentiation rules: chain, product, quotient, Implicit differentiation: differentiate both sides, then solve for dy/dx

Problems

1. Let $f(x) = x^3 + x + 1$. Explain why f has an inverse on all of $\mathbb{R}$, and find $(f^{-1})'(3)$ and $(f^{-1})'(1)$ without finding a formula for f^{-1}. difficulty ·

2. Let $f(x) = e^{2x} + x$. Find $(f^{-1})'(1)$ and the tangent line to the graph of f^{-1} at the point where $y = 1$. difficulty ·

3. Derive $(\arccos x)'$ from $\cos y = x$ and $(\arctan x)'$ from $\tan y = x$, explaining the sign of each square root. Evaluate both at $x = \frac{1}{2}$. difficulty ··

4. Differentiate $\arctan(3x)$, $\arcsin(x^2)$ and $\arctan(\sqrt{x})$. Evaluate them at $x = \frac{1}{3}$, $x = \frac{1}{2}$ and $x = 1$ respectively. difficulty ·

5. Show that $F(x) = x \arcsin x + \sqrt{1 - x^2}$ is an antiderivative of $\arcsin x$ on $(-1, 1)$, and use it to find $\displaystyle\int_0^{1/2} \arcsin x \, dx$. difficulty ··

6. The logit $\operatorname{logit}(p) = \ln \dfrac{p}{1-p}$ is the inverse of the sigmoid on $0 < p < 1$. Find its derivative twice: with the inverse function rule, and directly. Evaluate it at $p = 0.2$ and $p = 0.5$. difficulty ··

7. Let $g = f^{-1}$ for $f(x) = x + e^x$. Show that $g''(y) = -\dfrac{f''(x)}{f'(x)^3}$ at $x = g(y)$, and find $g'(1)$ and $g''(1)$. difficulty ···

8. Let $f(x) = x^2$ on $x \le 0$. Find a formula for f^{-1}, and compute $(f^{-1})'(4)$ both from the formula and with the inverse function rule. difficulty ··

9. A camera on the ground 500 m from a launch pad tracks a rocket rising straight up. When the rocket is 1000 m high it is climbing at 100 m/s. How fast, in radians per second, is the camera's angle of elevation changing? difficulty ···

10. A painting hangs with its bottom edge 1 m and its top edge 3 m above a viewer's eye level. Standing x m from the wall, the viewer sees it under the angle $\theta(x) = \arctan\dfrac{3}{x} - \arctan\dfrac{1}{x}$. Find the distance that makes θ largest, and the largest angle. difficulty ···

Answers

1. $(f^{-1})'(3) = \frac{1}{4}$ and $(f^{-1})'(1) = 1$
2. $(f^{-1})'(1) = \frac{1}{3}$; the tangent line to f^{-1} at $(1, 0)$ is $x = \frac{1}{3}(y - 1)$
3. $(\arccos x)' = -\dfrac{1}{\sqrt{1 - x^2}}$, which is $-\dfrac{2}{\sqrt{3}} \approx -1.1547$ at $\frac{1}{2}$; $(\arctan x)' = \dfrac{1}{1 + x^2}$, which is $\frac{4}{5}$ at $\frac{1}{2}$
4. $\dfrac{3}{1 + 9x^2} = \frac{3}{2}$ at $\frac{1}{3}$; $\dfrac{2x}{\sqrt{1 - x^4}} \approx 1.0328$ at $\frac{1}{2}$; $\dfrac{1}{2\sqrt{x}(1 + x)} = \frac{1}{4}$ at 1
5. $F' = \arcsin x$, and $\displaystyle\int_0^{1/2} \arcsin x \, dx = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1 \approx 0.1278$
6. $\operatorname{logit}'(p) = \dfrac{1}{p(1 - p)}$: 6.25 at $p = 0.2$ and 4 at $p = 0.5$
7. $g'(1) = \frac{1}{2}$ and $g''(1) = -\frac{1}{8}$
8. $f^{-1}(y) = -\sqrt{y}$ and $(f^{-1})'(4) = -\frac{1}{4}$, by both routes
9. $d\theta/dt = 0.04$ rad/s, about 2.3 degrees per second
10. Stand $\sqrt{3} \approx 1.73$ m away, where the angle is $\pi/6$, that is 30°

Worked solutions

Problem 1

> Let $f(x) = x^3 + x + 1$. Explain why f has an inverse on all of $\mathbb{R}$, and find $(f^{-1})'(3)$ and $(f^{-1})'(1)$ without finding a formula for f^{-1}.

$f'(x) = 3x^2 + 1 \geq 1 > 0$	Power rule; $3x^2 \geq 0$.
So f is increasing on $\mathbb{R}$, hence one-to-one, with an inverse	An increasing function never takes the same value twice.
$f(1) = 1 + 1 + 1 = 3$, so $f^{-1}(3) = 1$	Find the x that f sends to 3 by inspection.
$(f^{-1})'(3) = \dfrac{1}{f'(1)} = \dfrac{1}{4}$	The inverse function rule, with f' taken at $x = 1$, not at 3.
$f(0) = 1$, so $f^{-1}(1) = 0$ and $(f^{-1})'(1) = \dfrac{1}{f'(0)} = 1$	$f'(0) = 1$.
$(f^{-1})'(3) = \frac{1}{4}$ **and** $(f^{-1})'(1) = 1$	Solving $x^3 + x + 1 = y$ for x needs the cubic formula, but the slope of the inverse needs only one point of f and its slope there. A finite-difference slope of a numerical inverse agrees with both.

Problem 2

> Let $f(x) = e^{2x} + x$. Find $(f^{-1})'(1)$ and the tangent line to the graph of f^{-1} at the point where $y = 1$.

$f(0) = e^0 + 0 = 1$, so $f^{-1}(1) = 0$	The point $(0, 1)$ is on the graph of f, so $(1, 0)$ is on the graph of f^{-1}.
$f'(x) = 2e^{2x} + 1$, so $f'(0) = 3$	Chain rule on e^{2x}.

$(f^{-1})'(1) = \frac{1}{3}$	One over the slope of f at the matching point.
Tangent: $x = 0 + \frac{1}{3}(y - 1)$	Point-slope form for the inverse, written with y as its input.
$(f^{-1})'(1) = \frac{1}{3}$; the tangent line to f^{-1} at $(1, 0)$ is $x = \frac{1}{3}(y - 1)$	The tangent to f at $(0, 1)$ is $y = 1 + 3x$; solving it for x gives the same line, the reflection of f's tangent in $y = x$.

Problem 3

> Derive $(\arccos x)'$ from $\cos y = x$ and $(\arctan x)'$ from $\tan y = x$, explaining the sign of each square root. Evaluate both at $x = \frac{1}{2}$.

$y = \arccos x$ means $\cos y = x$ with $0 \le y \le \pi$	The principal branch.
$-\sin y \cdot y' = 1$, so $y' = -\dfrac{1}{\sin y}$	Differentiate both sides in x; the chain rule puts y' on the left.
$\sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - x^2}$	On $[0, \pi]$, $\sin y \ge 0$, so the positive root is the right one.
$(\arccos x)' = -\dfrac{1}{\sqrt{1 - x^2}}$ for $-1 < x < 1$	At $x = \pm 1$, $\sin y = 0$ and the slope is infinite.
$y = \arctan x$: $\sec^2 y \cdot y' = 1$	$(\tan y)' = \sec^2 y$.
$\sec^2 y = 1 + \tan^2 y = 1 + x^2$	The Pythagorean identity divided by $\cos^2 y$; no square root, so no sign to choose.
$(\arccos x)' = -\dfrac{1}{\sqrt{1 - x^2}}$, which is $-\dfrac{2}{\sqrt{3}} \approx -1.1547$ at $\frac{1}{2}$; $(\arctan x)' = \dfrac{1}{1 + x^2}$, which is $\frac{4}{5}$ at $\frac{1}{2}$	arccos falls from π to 0 as x goes from -1 to 1, so its slope must be negative. Its derivative is minus that of arcsin, because $\arcsin x + \arccos x = \frac{\pi}{2}$.

Problem 4

> Differentiate $\arctan(3x)$, $\arcsin(x^2)$ and $\arctan(\sqrt{x})$. Evaluate them at $x = \frac{1}{3}$, $x = \frac{1}{2}$ and $x = 1$ respectively.

$\dfrac{d}{dx}\arctan(3x) = \dfrac{1}{1+9x^2}\cdot 3$	Chain rule: the outer derivative at $u = 3x$, times $u' = 3$.
At $x = \frac{1}{3}$: $\dfrac{3}{1+1} = \frac{3}{2}$	$9 \cdot \frac{1}{9} = 1$.
$\dfrac{d}{dx}\arcsin(x^2) = \dfrac{2x}{\sqrt{1-x^4}}$	Outer derivative at $u = x^2$, so $1 - u^2 = 1 - x^4$; then $u' = 2x$.
At $x = \frac{1}{2}$: $\dfrac{1}{\sqrt{15/16}} = \dfrac{4}{\sqrt{15}} \approx 1.0328$	$2x = 1$ and $1 - \frac{1}{16} = \frac{15}{16}$.
$\dfrac{d}{dx}\arctan(\sqrt{x}) = \dfrac{1}{1+x}\cdot\dfrac{1}{2\sqrt{x}}$	$(\sqrt{x})^2 = x$ inside, then $(\sqrt{x})' = 1/(2\sqrt{x})$.
$\dfrac{3}{1+9x^2} = \frac{3}{2}$ **at** $\frac{1}{3}$; $\dfrac{2x}{\sqrt{1-x^4}} \approx 1.0328$ **at** $\frac{1}{2}$; $\dfrac{1}{2\sqrt{x}(1+x)} = \frac{1}{4}$ **at** 1	Each is the outer derivative evaluated at the inside, times the derivative of the inside. Finite differences agree with all three values.

Problem 5

> Show that $F(x) = x\arcsin x + \sqrt{1-x^2}$ is an antiderivative of $\arcsin x$ on $(-1, 1)$, and use it to find $\displaystyle\int_0^{1/2} \arcsin x\,dx$.

$(x\arcsin x)' = \arcsin x + \dfrac{x}{\sqrt{1-x^2}}$	Product rule.
$\left(\sqrt{1-x^2}\right)' = \dfrac{-2x}{2\sqrt{1-x^2}} = -\dfrac{x}{\sqrt{1-x^2}}$	Chain rule.
$F'(x) = \arcsin x$	The two fractions cancel.
$F(\frac{1}{2}) = \frac{1}{2}\cdot\frac{\pi}{6} + \frac{\sqrt{3}}{2}$ and $F(0) = 1$	$\arcsin\frac{1}{2} = \frac{\pi}{6}$ and $\sqrt{1-\frac{1}{4}} = \frac{\sqrt{3}}{2}$.

$F' = \arcsin x$, **and** $\displaystyle\int_0^{1/2} \arcsin x \, dx = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1 \approx 0.1278$	The fundamental theorem: $F(\frac{1}{2}) - F(0)$. The antiderivative comes from integration by parts with $u = \arcsin x$, $dv = dx$, and differentiating it is the check.

Problem 6

> The logit $\operatorname{logit}(p) = \ln \dfrac{p}{1-p}$ is the inverse of the sigmoid on $0 < p < 1$. Find its derivative twice: with the inverse function rule, and directly. Evaluate it at $p = 0.2$ and $p = 0.5$.

$\sigma(\operatorname{logit}(p)) = p$	$\sigma(t) = p$ solves to $e^{-t} = (1-p)/p$, that is $t = \ln \frac{p}{1-p}$.
$\operatorname{logit}'(p) = \dfrac{1}{\sigma'(t)}$ with $t = \operatorname{logit}(p)$	The inverse function rule.
$\sigma'(t) = \sigma(t)(1 - \sigma(t)) = p(1-p)$	At the matching point, $\sigma(t) = p$.
Directly: $\dfrac{d}{dp}\big(\ln p - \ln(1-p)\big) = \dfrac{1}{p} + \dfrac{1}{1-p}$	Split the log of a quotient.
$\dfrac{1}{p} + \dfrac{1}{1-p} = \dfrac{1-p+p}{p(1-p)}$	Common denominator: the two routes agree.
$\operatorname{logit}'(p) = \dfrac{1}{p(1-p)}$: 6.25 **at** $p = 0.2$ **and** 4 **at** $p = 0.5$	The sigmoid is flattest near 0 and 1, so its inverse is steepest there: near a probability of 0 or 1, a small change in p is a large change in log-odds.

Problem 7

> Let $g = f^{-1}$ for $f(x) = x + e^x$. Show that $g''(y) = -\dfrac{f''(x)}{f'(x)^3}$ at $x = g(y)$, and find $g'(1)$ and $g''(1)$.

$g'(y) = \dfrac{1}{f'(g(y))}$	The inverse function rule, written as a function of y.

$g''(y) = -\dfrac{f''(g(y))\,g'(y)}{f'(g(y))^2}$	Differentiate $1/u$ with $u = f'(g(y))$: $(1/u)' = -u'/u^2$, and $u' = f''(g(y))\,g'(y)$ by the chain rule.
$g''(y) = -\dfrac{f''(x)}{f'(x)^3}$	Substitute $g'(y) = 1/f'(x)$ from step 1.
$f(0) = 1$, so $g(1) = 0$; $f'(x) = 1 + e^x$ and $f''(x) = e^x$	$f'(0) = 2$, $f''(0) = 1$.
$g'(1) = \frac{1}{2}$ **and** $g''(1) = -\frac{1}{8}$	f is convex and increasing, so its reflection is concave, and $g'' < 0$ agrees. The inverse here has no elementary formula (it needs the Lambert W function), and a second difference of a numerical inverse gives -0.125.

Problem 8

> Let $f(x) = x^2$ on $x \le 0$. Find a formula for f^{-1}, and compute $(f^{-1})'(4)$ both from the formula and with the inverse function rule.

f is decreasing on $(-\infty, 0]$, with range $[0, \infty)$	$f'(x) = 2x < 0$ for $x < 0$.
$y = x^2$ with $x \le 0$ gives $x = -\sqrt{y}$, so $f^{-1}(y) = -\sqrt{y}$	Of the two square roots, the branch with $x \le 0$ is the negative one.
From the formula: $(f^{-1})'(y) = -\dfrac{1}{2\sqrt{y}}$, so $-\frac{1}{4}$ at $y = 4$	Power rule.
With the rule: $f^{-1}(4) = -2$ and $f'(-2) = -4$	The matching point is $x = -2$, not $x = 2$.

$f^{-1}(y) = -\sqrt{y}$ **and** $(f^{-1})'(4) = -\frac{1}{4}$**, by both routes**	The inverse of a decreasing function is decreasing. The point $x = 2$ is not in the domain of this f, and using it gives $+\frac{1}{4}$ (Mistake 5).

Problem 9

A camera on the ground 500 m from a launch pad tracks a rocket rising straight up. When the rocket is 1000 m high it is climbing at 100 m/s. How fast, in radians per second, is the camera's angle of elevation changing?

$\theta = \arctan \dfrac{h}{500}$	The right triangle with legs h (up) and 500 (along the ground).
$\dfrac{d\theta}{dt} = \dfrac{1}{1 + (h/500)^2} \cdot \dfrac{1}{500} \dfrac{dh}{dt}$	Chain rule through $\arctan$, then through $h/500$, then through $h(t)$.
At $h = 1000$: $\dfrac{1}{1+4} = \frac{1}{5}$	$h/500 = 2$.
$\frac{1}{5} \cdot \frac{100}{500} = \frac{1}{25}$	$dh/dt = 100$.
$d\theta/dt = 0.04$ **rad/s, about** 2.3 **degrees per second**	The angle changes more slowly the higher the rocket goes: the factor $1/(1 + (h/500)^2)$ shrinks as h grows, though the rocket's speed is constant.

Problem 10

A painting hangs with its bottom edge 1 m and its top edge 3 m above a viewer's eye level. Standing x m from the wall, the viewer sees it under the angle $\theta(x) = \arctan \dfrac{3}{x} - \arctan \dfrac{1}{x}$. Find the distance that makes θ largest, and the largest angle.

$\dfrac{d}{dx} \arctan \dfrac{a}{x} = \dfrac{1}{1 + a^2/x^2} \cdot \left(-\dfrac{a}{x^2} \right) = -\dfrac{a}{x^2 + a^2}$	Chain rule, then multiply top and bottom by x^2.
$\theta'(x) = -\dfrac{3}{x^2 + 9} + \dfrac{1}{x^2 + 1}$	Step 1 with $a = 3$ and $a = 1$.

$\theta' = 0$ when $x^2 + 9 = 3(x^2 + 1)$	Cross-multiply.
$2x^2 = 6$, so $x = \sqrt{3} \approx 1.732$	Only the positive root is a distance.
$\theta'(1) = 0.2 > 0$ and $\theta'(3) \approx -0.067 < 0$	The angle rises, then falls: a maximum.
$\theta(\sqrt{3}) = \arctan\sqrt{3} - \arctan\frac{1}{\sqrt{3}} = \frac{\pi}{3} - \frac{\pi}{6}$	$\tan\frac{\pi}{3} = \sqrt{3}$ and $\tan\frac{\pi}{6} = 1/\sqrt{3}$.
Stand $\sqrt{3} \approx 1.73$ m away, where the angle is $\pi/6$, that is 30°	Close up the painting is seen almost edge on, and far away it shrinks; in between, the best distance is the geometric mean $\sqrt{1 \cdot 3}$ of the two heights.

Where this goes wrong

1. Reading f⁻¹ as 1/f

The superscript -1 also means a reciprocal, as in $x^{-1} = 1/x$.

$f(x) = x^3 + x + 1$; find $(f^{-1})'(3)$	Right so far: Problem 1.
"$f^{-1} = 1/f$, so differentiate $1/f$ and put in 3."	The habit that causes the mistake: the exponent rule carried over to function notation. The reciprocal of f is written $1/f$ or $f(x)^{-1}$.
$(f^{-1})'(3) = -\dfrac{f'(3)}{f(3)^2} = -\dfrac{28}{961} \approx -0.0291$ ✗	That is the slope of $1/f$ at $x = 3$. The inverse is increasing, so its slope cannot be negative; the right value is $\frac{1}{4}$.

2. Evaluating f′ at y instead of at f⁻¹(y)

The rule has an f' in the denominator and a 3 in the question, and the 3 goes straight in.

$(f^{-1})'(3) = \dfrac{1}{f'(\,\cdot\,)}$ with $f'(x) = 3x^2 + 1$	Right so far.

“The slope at 3 is one over $f'(3)$.”	The slip that causes the mistake: 3 is a value of y. The slope of f must be taken at the x with $f(x) = 3$, which is $x = 1$.
$(f^{-1})'(3) = \dfrac{1}{f'(3)} = \dfrac{1}{28} \approx 0.0357$ ✗	Seven times too small. The two coordinates swap under reflection in $y = x$, so the point $(3, 1)$ on the inverse matches $(1, 3)$ on f.

3. Giving arccos a positive slope

The derivation of $(\arcsin x)'$ is fresh, and arccos looks like the same function with a different name.

$\cos y = x$, so $-\sin y \cdot y' = 1$	Right so far: Problem 3, step 2.
“The minus sign cancels against the square root, as for arcsin.”	The habit that causes the mistake: the arcsin result copied. Here $\sin y = \sqrt{1 - x^2}$ is positive, so the minus stays.
$(\arccos x)' = \dfrac{1}{\sqrt{1 - x^2}}$, so 1.1547 at $\frac{1}{2}$ ✗	arccos goes from π at $x = -1$ down to 0 at $x = 1$, so it is decreasing and its slope at $\frac{1}{2}$ is -1.1547.

4. Dropping the inner derivative

The derivative of arctan is memorized as $1/(1 + x^2)$, and the x is replaced by whatever is inside.

$\dfrac{d}{dx} \arctan(3x)$ at $x = \frac{1}{3}$	Right so far: Problem 4.
“Replace x with $3x$ in $\dfrac{1}{1 + x^2}$.”	Right as far as it goes: that is the outer derivative at $u = 3x$. The chain rule still needs the factor $u' = 3$.
$\dfrac{1}{1 + 9x^2} = \frac{1}{2}$ at $x = \frac{1}{3}$ ✗	The slope is $\frac{3}{2}$. $\arctan(3x)$ is arctan squeezed horizontally by 3, so it rises three times as fast at the matching point.

5. Choosing the wrong branch

The inverse function rule asks for the x with $f(x) = 4$, and $x = 2$ is the first one that comes to mind.

$f(x) = x^2$ on $x \le 0$; find $(f^{-1})'(4)$	Right so far: Problem 8.
"$f(2) = 4$, so the matching point is $x = 2$."	The slip that causes the mistake: 2 is not in the domain of this f. On $x \le 0$ the only solution of $x^2 = 4$ is $x = -2$.
$(f^{-1})'(4) = \dfrac{1}{f'(2)} = \frac{1}{4}$ ✗	It should be $\dfrac{1}{f'(-2)} = -\frac{1}{4}$. A function and its inverse are either both increasing or both decreasing, and x^2 on $x \le 0$ is decreasing.

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