

Integration: substitution and parts

Ten integrals by substitution and by parts — including parts applied twice, a definite integral, an improper one, and the sigmoid whose antiderivative is softplus — with worked solutions and the mistakes that forget the du or pick u and dv the wrong way round.

BEFORE YOU START

Two techniques cover every integral on this page: substitution, which undoes the chain rule, and integration by parts, which undoes the product rule. The work goes wrong in a few places that recur: a dx replaced without the factor that du brings, u and dv chosen the wrong way round, a sign lost from du , limits left unchanged after a substitution, and a value at infinity asserted instead of computed. These ten problems run from single substitutions and single applications of parts to parts applied twice, a definite integral, an improper one, and the integral of the sigmoid.

- $\int f dx = F + C$ means $F' = f$: F is an antiderivative of f , and the constant C stands for any constant, since adding a constant does not change the derivative.
- Substitution: set $u = g(x)$, so $du = g'(x) dx$, and rewrite the whole integrand, dx included, in terms of u . In a definite integral, change the limits with it: $x = a$ becomes $u = g(a)$.
- Integration by parts: $\int u dv = uv - \int v du$. It is the product rule $(uv)' = u'v + uv'$ integrated and rearranged. For a definite integral, $\int_a^b u dv = [uv]_a^b - \int_a^b v du$, where $[F]_a^b = F(b) - F(a)$.
- u has a meaning local to each problem: in a substitution it is the new variable, in parts it is the factor to be differentiated, with dv the part to be integrated. No problem uses both at once.
- An improper integral is a limit: $\int_0^\infty f dx$ means $\lim_{b \rightarrow \infty} \int_0^b f dx$.
- $\ln$ is the natural log, defined for positive arguments; $\int \frac{du}{u} = \ln |u| + C$ on any interval where $u \neq 0$.
- Every antiderivative below can be checked by differentiating it — do that before looking at the solution.

Builds on: Differentiation rules: chain, product, quotient

Problems

1. $\int x e^{x^2} dx$

difficulty ·

2. $\int \frac{2x}{1+x^2} dx$

difficulty ·

3. $\int x e^x dx$

difficulty ··

4. $\int \ln x dx$

difficulty ··

5. $\int x \cos x dx$

difficulty ··

6. $\int e^x \sin x dx$

difficulty ...

7. $\int_0^1 \frac{dx}{1+x^2}$

difficulty ··

8. $\int \tan x dx$

difficulty ...

9. $\int_0^\infty x e^{-x} dx$

difficulty ...

10. $\int \sigma(x) dx$ where $\sigma(x) = \frac{1}{1+e^{-x}}$, and then $\int_{-\infty}^0 \sigma(x) dx$.

difficulty ...

Answers

1. $\frac{1}{2}e^{x^2} + C$

2. $\ln(1 + x^2) + C$

3. $(x - 1)e^x + C$

4. $x \ln x - x + C$

5. $x \sin x + \cos x + C$

6. $\frac{1}{2}e^x(\sin x - \cos x) + C$

7. $\frac{\pi}{4}$

8. $-\ln |\cos x| + C$

9. 1

10. $\ln(1 + e^x) + C$; and $\int_{-\infty}^0 \sigma = \ln 2 - \lim_{a \rightarrow -\infty} \ln(1 + e^a) = \ln 2$

Worked solutions

Problem 1

$$\int x e^{x^2} dx$$

$$u = x^2, du = 2x dx.$$

e^{x^2} is e^u with inner function x^2 , and the integrand also contains x , which is half of that inner function's derivative $2x$.

$$x dx = \frac{1}{2} du.$$

Divide both sides of step 1 by 2, so that the $x dx$ in the integral can be replaced as a whole.

$$\int x e^{x^2} dx = \int e^u \cdot \frac{1}{2} du = \frac{1}{2} e^u + C.$$

Substitute steps 1 and 2; $(e^u)' = e^u$, and the constant $\frac{1}{2}$ comes out of the integral.

$$\frac{1}{2} e^{x^2} + C$$

Put $u = x^2$ back. Check: by the chain rule, $\frac{d}{dx} \frac{1}{2} e^{x^2} = \frac{1}{2} e^{x^2} \cdot 2x = x e^{x^2}$.

Problem 2

$$\int \frac{2x}{1+x^2} dx$$

$$u = 1 + x^2, du = 2x dx.$$

The numerator $2x dx$ is exactly the derivative of the denominator, times dx .

$$\int \frac{2x}{1+x^2} dx = \int \frac{du}{u} = \ln |u| + C.$$

The denominator becomes u and the numerator with dx becomes du .

$$\ln(1+x^2) + C$$

Put $u = 1 + x^2$ back; $1 + x^2 \geq 1 > 0$ for every x , so the absolute value can be dropped and the answer holds on the whole real line.

Problem 3

$$\int x e^x dx$$

$$u = x, dv = e^x dx; du = dx, v = e^x.$$

Differentiating x makes it simpler, the constant 1, and e^x is its own antiderivative, so the new integral $\int v du$ has no factor of x left.

$$\int x e^x dx = x e^x - \int e^x dx.$$

$$\text{Parts: } uv - \int v du.$$

$$x e^x - e^x + C.$$

$$\int e^x dx = e^x + C.$$

$$(x - 1)e^x + C$$

Factor out e^x . Check: by the product rule, $((x - 1)e^x)' = e^x + (x - 1)e^x = x e^x$.

Problem 4

$$\int \ln x dx$$

$$u = \ln x, dv = dx; du = \frac{dx}{x}, v = x.$$

No antiderivative of $\ln x$ is known yet, but its derivative, $1/x$, is simple; the other factor is then 1, and dx integrates to x . $\ln x$ needs $x > 0$.

$$\int \ln x dx = x \ln x - \int x \cdot \frac{1}{x} dx.$$

$$\text{Parts: } uv - \int v du.$$

$$x \ln x - \int 1 dx = x \ln x - x + C.$$

$$x \cdot \frac{1}{x} = 1 \text{ for } x > 0, \text{ and } \int 1 dx = x + C.$$

$$x \ln x - x + C$$

Check: $(x \ln x - x)' = \ln x + x \cdot \frac{1}{x} - 1 = \ln x$.

Problem 5

$$\int x \cos x \, dx$$

$$u = x, dv = \cos x \, dx; du = dx, v = \sin x.$$

Differentiating x removes it, and $\cos x$ integrates to $\sin x$ without getting harder.

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx.$$

Parts: $uv - \int v \, du$.

$$\int \sin x \, dx = -\cos x + C.$$

$$(-\cos x)' = \sin x.$$

$$x \sin x + \cos x + C$$

Subtracting $-\cos x$ adds $\cos x$. Check:
 $(x \sin x + \cos x)' = \sin x + x \cos x - \sin x = x \cos x$.

Problem 6

$$\int e^x \sin x \, dx$$

$$\text{Call the integral } I = \int e^x \sin x \, dx.$$

Neither factor gets simpler when differentiated, so parts will not finish the job; the plan is to apply parts twice until I reappears, then solve for it.

$$u = \sin x, dv = e^x \, dx; du = \cos x \, dx, v = e^x.$$

e^x integrates to itself, so v stays as simple as dv .

$$I = e^x \sin x - \int e^x \cos x \, dx.$$

Parts: $uv - \int v \, du$.

$$\text{For } \int e^x \cos x \, dx: u = \cos x, dv = e^x \, dx; \\ du = -\sin x \, dx, v = e^x.$$

Keep the same choice, trig as u and the exponential as dv . Swapping it would integrate the $\cos x$ back to $\sin x$ and undo step 3, giving only $I = I$.

$\int e^x \cos x \, dx = e^x \cos x -$ $\int e^x (-\sin x) \, dx = e^x \cos x + I.$	Parts again; the minus from du turns the new integral into $+I$.
$I = e^x \sin x - e^x \cos x - I.$	Substitute step 5 into step 3. The original integral now appears on both sides.
$2I = e^x (\sin x - \cos x).$	Add I to both sides and factor out e^x . The constants of integration from the two applications of parts are gathered into one C in the final line.
$\frac{1}{2}e^x (\sin x - \cos x) + C$	Divide by 2. Check: $(\frac{1}{2}e^x (\sin x - \cos x))' = \frac{1}{2}e^x (\sin x - \cos x) + \frac{1}{2}e^x (\cos x + \sin x) = e^x \sin x$.

Problem 7

$\int_0^1 \frac{dx}{1+x^2}$	
$y = \arctan x$ means $\tan y = x$ with $-\frac{\pi}{2} < y < \frac{\pi}{2}$.	$\arctan$ is the inverse of $\tan$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$, where $\cos y > 0$ and $\tan$ is increasing, so each x has exactly one y .
$\frac{1}{\cos^2 y} y' = 1.$	Differentiate $\tan y = x$ with respect to x : the chain rule gives the derivative of $\tan$ at y , which is $1/\cos^2 y$ (quotient rule on $\sin y/\cos y$), times y' .
$y' = \cos^2 y = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}.$	$1 + \tan^2 y = \frac{\cos^2 y + \sin^2 y}{\cos^2 y} = \frac{1}{\cos^2 y},$ and $\tan y = x$.
$\int_0^1 \frac{dx}{1+x^2} = [\arctan x]_0^1 = \arctan 1 - \arctan 0.$	Step 3 makes $\arctan x$ an antiderivative of the integrand, and a definite integral is the antiderivative's value at the upper limit minus its value at the lower one.
$\arctan 1 = \frac{\pi}{4} \text{ and } \arctan 0 = 0.$	$\tan \frac{\pi}{4} = 1$ and $\tan 0 = 0$, and both angles lie in $(-\frac{\pi}{2}, \frac{\pi}{2})$.

$$\frac{\pi}{4}$$

$\frac{\pi}{4} - 0$. The integrand lies between $\frac{1}{2}$ and 1 on $[0, 1]$, and $\frac{\pi}{4} \approx 0.785$ is between them.

Problem 8

$$\int \tan x \, dx$$

$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$, on an interval where $\cos x \neq 0$.

$\tan x$ is defined only where $\cos x \neq 0$; written as a quotient, the numerator is, up to sign, the derivative of the denominator.

$u = \cos x$, $du = -\sin x \, dx$, so $\sin x \, dx = -du$.

$(\cos x)' = -\sin x$; the minus sign has to be carried into the integral.

$$\int \frac{\sin x}{\cos x} \, dx = \int \frac{-du}{u} = -\ln |u| + C.$$

$\int du/u = \ln |u| + C$ for $u \neq 0$; the absolute value is needed because $\cos x$ is negative on half of the intervals where $\tan x$ is defined.

$$-\ln |\cos x| + C$$

Put $u = \cos x$ back. Check: where $\cos x > 0$, the chain rule gives $\frac{d}{dx} \ln \cos x = \frac{-\sin x}{\cos x} = -\tan x$; where $\cos x < 0$, it gives $\frac{d}{dx} \ln(-\cos x) = \frac{\sin x}{-\cos x} = -\tan x$. Either way the minus sign in front makes the derivative $\tan x$.

Problem 9

$$\int_0^{\infty} x e^{-x} \, dx$$

$$\int_0^{\infty} x e^{-x} \, dx = \lim_{b \rightarrow \infty} \int_0^b x e^{-x} \, dx.$$

The upper limit is not a number, so the integral is defined as the limit of integrals over $[0, b]$.

$$u = x, \, dv = e^{-x} \, dx; \, du = dx, \, v = -e^{-x}.$$

Differentiating x removes it; $(-e^{-x})' = e^{-x}$ by the chain rule.

$\int_0^b x e^{-x} dx = [-x e^{-x}]_0^b + \int_0^b e^{-x} dx.$	Parts on $[0, b]$: $[uv]_0^b - \int_0^b v du$, and $-v = e^{-x}$.
$[-x e^{-x}]_0^b = -b e^{-b} - 0 \text{ and } \int_0^b e^{-x} dx = [-e^{-x}]_0^b = 1 - e^{-b}.$	Evaluate each bracket at b and at 0 ; at $x = 0$ the first bracket is $0 \cdot e^0 = 0$.
$\int_0^b x e^{-x} dx = 1 - e^{-b} - b e^{-b}.$	Add the two parts of step 4.
$0 < b e^{-b} = \frac{b}{e^b} < \frac{b}{b^2/2} = \frac{2}{b} \text{ for } b > 0, \text{ so } b e^{-b} \rightarrow 0.$	$e^b = 1 + b + \frac{b^2}{2} + \dots > \frac{b^2}{2}$ for $b > 0$, since every term of the series is positive; the bound $2/b$ goes to 0 .
1	Take $b \rightarrow \infty$ in step 5: $e^{-b} \rightarrow 0$ and, by step 6, $b e^{-b} \rightarrow 0$. This is the mean of the exponential distribution with rate 1.

Problem 10

$\int \sigma(x) dx$ where $\sigma(x) = \frac{1}{1 + e^{-x}}$, and then $\int_{-\infty}^0 \sigma(x) dx$.

$\sigma(x) = \frac{e^x}{1 + e^x}.$	Multiply top and bottom by e^x , which is never 0 , so that the numerator becomes the derivative of the denominator.
$u = 1 + e^x, du = e^x dx.$	The denominator's derivative, e^x , is the numerator.
$\int \frac{e^x}{1 + e^x} dx = \int \frac{du}{u} = \ln u + C.$	Substitute step 2.
$\int \sigma(x) dx = \ln(1 + e^x) + C.$	Put $u = 1 + e^x$ back; $1 + e^x > 1$, so the absolute value can be dropped. This is softplus, whose derivative is σ (Differentiation rules, Problem 4).

$\int_{-\infty}^0 \sigma(x) dx = \lim_{a \rightarrow -\infty} \int_a^0 \sigma(x) dx = \lim_{a \rightarrow -\infty} (\ln 2 - \ln(1 + e^a)).$	The lower limit is not a number, so the integral is a limit; step 4 evaluated at 0 gives $\ln 2$, since $1 + e^0 = 2$, and at a gives $\ln(1 + e^a)$.
$\ln(1 + e^a) \rightarrow \ln 1 = 0 \text{ as } a \rightarrow -\infty.$	$e^a \rightarrow 0, \text{ and } \ln \text{ is continuous at } 1.$
$\ln(1 + e^x) + C; \text{ and } \int_{-\infty}^0 \sigma = \ln 2 - \lim_{a \rightarrow -\infty} \ln(1 + e^a) = \ln 2$	Steps 4 to 6. The area is finite although σ never reaches 0, because $\sigma(x) < e^x$ decays exponentially as $x \rightarrow -\infty$.

Where this goes wrong

1. Substituting without changing dx

Setting $u = x^2$ turns e^{x^2} into e^u at a glance, and the rest of the integrand is easy to wave through.

$\int x e^{x^2} dx \text{ with } u = x^2$	Right so far: the substitution of Problem 1.
“e^{x^2} becomes e^u, and $x dx$ becomes du.”	The shortcut that causes the mistake: $x dx$ treated as the whole of du.
$\int x e^{x^2} dx = \int e^u du = e^{x^2} + C \quad \times$	$du = 2x dx$, so $x dx = \frac{1}{2} du$, and the factor $\frac{1}{2}$ is lost. Differentiating the answer shows it: $(e^{x^2})' = 2x e^{x^2}$, twice the integrand. The correct antiderivative is $\frac{1}{2} e^{x^2} + C$.

2. Parts with u and dv swapped

Parts works with either choice of u and dv in the sense that the formula stays true, so it is tempting to take the first factor that integrates easily.

$\int x e^x dx \text{ by parts}$	Right so far: a product of a polynomial and an exponential, the case parts is for.
“x integrates easily too, so either factor can be dv.”	The reasoning that causes the mistake: choosing dv by what can be integrated, without asking what the new integral will look like.

$$u = e^x, dv = x dx \text{ in } \int x e^x dx \text{ ✗}$$

This gives $v = \frac{x^2}{2}$, $du = e^x dx$ and $\int x e^x dx = \frac{x^2}{2} e^x - \int \frac{x^2}{2} e^x dx$ — true, but the new integral is harder than the old one, which is the sign that the choice is wrong. Pick u to be what gets simpler when differentiated: $u = x$ gives $(x - 1)e^x + C$ (Problem 3).

3. Sign error in $\int \tan$

$\tan x = \sin x / \cos x$ looks like a derivative over a function, and $\int u'/u = \ln |u|$ is the pattern that comes to mind.

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx \text{ with } u = \cos x$$

Right so far: the substitution of Problem 8.

“The numerator is the derivative of the denominator, so the answer is $\ln$ of the denominator.”

The pattern that causes the mistake: $\sin x$ taken as the derivative of $\cos x$.

$$\int \tan x dx = \ln |\cos x| + C \text{ ✗}$$

$du = -\sin x dx$ brings a minus: $\sin x dx = -du$, so the integral is $-\ln |\cos x| + C$. Differentiating this line shows it: $(\ln |\cos x|)' = -\tan x$.

4. Changing the variable but keeping the old limits

A substitution in an indefinite integral changes only the integrand, and the habit carries over to definite integrals.

$$\int_0^1 \frac{2x}{1+x^2} dx \text{ with } u = 1+x^2, du = 2x dx$$

Right so far: the substitution of Problem 2.

“The integrand becomes du/u ; the rest stays.”

The habit that causes the mistake: the limits, which are values of x , left in place.

$$\int_0^1 \frac{2x}{1+x^2} dx = \int_0^1 \frac{du}{u} \quad \times$$

The limits must be values of u : $u(0) = 1$ and $u(1) = 2$, so the integral is $\int_1^2 du/u = \ln 2$. With the old limits the integral diverges, because $1/u$ is not integrable near $u = 0$ — yet the original integrand is continuous and bounded on $[0, 1]$, so its integral is finite.

5. Evaluating $\infty \cdot 0$ by feel

In Problem 9 the boundary term at infinity is x , which grows, times e^{-x} , which shrinks, and it is tempting to call the product 0 and move on.

$$\int_0^\infty x e^{-x} dx = [-x e^{-x}]_0^\infty + \int_0^\infty e^{-x} dx$$

Right so far as a plan: parts with $u = x$, $dv = e^{-x} dx$, as in Problem 9.

“At infinity e^{-x} is 0, so $-x e^{-x}$ is 0 there.”

The reasoning that causes the mistake: $\infty \cdot 0$ read as 0.

$$[-x e^{-x}]_0^\infty = 0 - 0 \quad \times$$

The value happens to be right here, because $b e^{-b} \rightarrow 0$ (Problem 9, step 6), but the line asserts it without a limit, and $\infty \cdot 0$ is not a value: $b \cdot \frac{1}{b} \rightarrow 1$, while $b \cdot b^{-1/2} \rightarrow \infty$. The same shortcut on $\int_1^\infty x \cdot \frac{1}{2} x^{-3/2} dx$, by parts with $u = x$ and $v = -x^{-1/2}$, calls the boundary term $[-x \cdot x^{-1/2}]_1^\infty$ zero at infinity, but at b it is $-\sqrt{b} \rightarrow -\infty$, and that integral diverges. Writing $\lim_{b \rightarrow \infty}$ is where the answer and the convergence are decided.