

Implicit differentiation: differentiate both sides, then solve for dy/dx

Ten implicit derivatives: the slope and tangent line of a circle and an ellipse, a curve with an xy term, an exponential and a log, a trig relation, the folium of Descartes, the horizontal and vertical tangents of a tilted ellipse, a second derivative, the derivative of arcsin, and logarithmic differentiation of a product and quotient, with worked solutions and the mistakes of dropping dy/dx , skipping the product rule, differentiating a constant, losing a y' term and treating y as a constant.

BEFORE YOU START

Some curves are given by an equation in x and y that is hard or impossible to solve for y , such as $x^3 + y^3 = 6xy$. Implicit differentiation finds the slope anyway: treat y as a function of x , differentiate both sides of the equation with respect to x , and solve the result for y' . It is the same chain rule that drives related rates, and the idea behind differentiating through a quantity defined by an equation, such as the optimum of an inner problem, instead of by a formula. The work goes wrong in a few places: a y' dropped from the chain rule, a product rule skipped on xy , a constant side differentiated as if it were a variable, a y' term left on the wrong side, and y treated as a constant in a second derivative. These ten problems cover each of them.

- y is a function of x , and y' means $\frac{dy}{dx}$. Every term containing y is a composition, so its derivative carries a factor y' : $\frac{d}{dx} y^2 = 2y y'$, $\frac{d}{dx} \sin y = \cos y y'$, $\frac{d}{dx} e^y = e^y y'$, $\frac{d}{dx} \ln y = \frac{y'}{y}$.
- A term with both variables needs the product rule as well: $\frac{d}{dx}(xy) = y + x y'$. A constant differentiates to 0.
- The method: (1) differentiate both sides of the equation with respect to x ; (2) move every term containing y' to one side and everything else to the other; (3) factor out y' and divide. Substitute the point's coordinates at the end, or after step 1 if only a number is wanted.
- The slope usually depends on both x and y . The tangent line at (x_0, y_0) is $y - y_0 = m(x - x_0)$ with $m = y'$ there. A tangent is horizontal where $y' = 0$ and vertical where the denominator of y' is 0 and the numerator is not.
- Angles are in radians.

Builds on: Differentiation rules: chain, product, quotient

Problems

1. Find y' for the circle $x^2 + y^2 = 25$, and the tangent line at $(3, 4)$. difficulty ·

2. Find y' for the ellipse $x^2 + 4y^2 = 8$, and its value at $(2, 1)$ and at $(2, -1)$. difficulty ·

3. Find y' for $x^2 + 3xy + y^2 = 11$ at $(1, 2)$. difficulty ··

4. Find y' for $ye^x + \ln y = 1$ at the point where $x = 0$. difficulty ··

5. Find y' for $x \sin y + y = \pi$ at $(\frac{\pi}{2}, \frac{\pi}{2})$. difficulty ··

6. The folium of Descartes is $x^3 + y^3 = 6xy$. Find the slope and the tangent line at $(3, 3)$. difficulty ···

7. Find the points on $x^2 - xy + y^2 = 3$ where the tangent is horizontal, and where it is vertical. difficulty ···

8. For $x^2 + y^2 = 25$, find y'' in terms of y alone, and its value at $(3, 4)$. difficulty ···

9. Use $\sin y = x$ to find the derivative of $y = \arcsin x$, and its value at $x = \frac{1}{2}$. difficulty ··

10. Use logarithmic differentiation to find y' for $y = \frac{(x+1)^2(x+2)^3}{(x+3)^4}$, and its value at $x = 0$. difficulty ···

Answers

1. $y' = -\frac{x}{y}$; tangent $y = -\frac{3}{4}x + \frac{25}{4}$

2. $y' = -\frac{x}{4y}$; $-\frac{1}{2}$ at $(2, 1)$ and $\frac{1}{2}$ at $(2, -1)$

3. $y' = -\frac{2+6}{3+4} = -\frac{8}{7}$

4. $y' = -\frac{1}{2}$

5. $y' = -\frac{1}{0+1} = -1$

6. Slope -1 ; tangent $y = -x + 6$

7. Horizontal at $(1, 2)$ and $(-1, -2)$; vertical at $(2, 1)$ and $(-2, -1)$

8. $y'' = -\frac{25}{y^3}$; at $(3, 4)$, $-\frac{25}{64} \approx -0.391$

9. At $x = \frac{1}{2}$: $\frac{1}{\sqrt{3/4}} = \frac{2}{\sqrt{3}} \approx 1.155$

10. $y' = \frac{8}{81} \cdot \frac{13}{6} = \frac{52}{243} \approx 0.214$

Worked solutions

Problem 1

Find y' for the circle $x^2 + y^2 = 25$, and the tangent line at $(3, 4)$.

$2x + 2y y' = 0$	Differentiate both sides; $\frac{d}{dx} y^2 = 2y y'$ by the chain rule, and 25 gives 0.
$y' = -\frac{x}{y}$	Move $2x$ across and divide by $2y$.
At $(3, 4)$: $y' = -\frac{3}{4}$	Substitute.
$y - 4 = -\frac{3}{4}(x - 3)$	Point-slope form.
$y' = -\frac{x}{y}$; tangent $y = -\frac{3}{4}x + \frac{25}{4}$	The tangent is perpendicular to the radius, whose slope is $\frac{4}{3}$.

Problem 2

Find y' for the ellipse $x^2 + 4y^2 = 8$, and its value at $(2, 1)$ and at $(2, -1)$.

$2x + 8y y' = 0$	Differentiate both sides.
$y' = -\frac{x}{4y}$	Solve for y' .
At $(2, 1)$: $y' = -\frac{2}{4} = -\frac{1}{2}$. At $(2, -1)$: $y' = \frac{1}{2}$.	Same x , two points on the curve.
$y' = -\frac{x}{4y}$; $-\frac{1}{2}$ at $(2, 1)$ and $\frac{1}{2}$ at $(2, -1)$	The slope needs both coordinates, because the ellipse is not the graph of one function of x .

Problem 3

Find y' for $x^2 + 3xy + y^2 = 11$ at $(1, 2)$.

$1 + 6 + 4 = 11$	The point is on the curve.
$2x + (3y + 3x y') + 2y y' = 0$	The product rule on $3xy$ gives $3y + 3x y'$.
$y'(3x + 2y) = -(2x + 3y)$	Collect the y' terms on one side.
$y' = -\frac{2x + 3y}{3x + 2y}$	Divide.
$y' = -\frac{2 + 6}{3 + 4} = -\frac{8}{7}$	Substitute $x = 1, y = 2$.

Problem 4

Find y' for $y e^x + \ln y = 1$ at the point where $x = 0$.

At $x = 0$: $y + \ln y = 1$, so $y = 1$.	$1 + \ln 1 = 1$, and $y + \ln y$ is increasing, so 1 is the only solution.
$y' e^x + y e^x + \frac{y'}{y} = 0$	Product rule on $y e^x$; chain rule on $\ln y$.
$y' \left(e^x + \frac{1}{y} \right) = -y e^x$	Collect the y' terms.
$y'(1 + 1) = -1$	Substitute $x = 0, y = 1$.
$y' = -\frac{1}{2}$	Solving for y first is not possible in elementary functions, and it was never needed.

Problem 5

Find y' for $x \sin y + y = \pi$ at $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$\frac{\pi}{2} \cdot 1 + \frac{\pi}{2} = \pi$	The point is on the curve.
---	----------------------------

$\sin y + x \cos y y' + y' = 0$	Product rule on $x \sin y$, with $\frac{d}{dx} \sin y = \cos y y'$; the constant π gives 0.
$y' = -\frac{\sin y}{x \cos y + 1}$	Collect and divide.
$y' = -\frac{1}{0 + 1} = -1$	$\sin \frac{\pi}{2} = 1$ and $\cos \frac{\pi}{2} = 0$.

Problem 6

The folium of Descartes is $x^3 + y^3 = 6xy$. Find the slope and the tangent line at $(3, 3)$.

$27 + 27 = 54 = 6 \cdot 9$	The point is on the curve.
$3x^2 + 3y^2 y' = 6y + 6x y'$	Chain rule on y^3 ; product rule on the right.
$y'(3y^2 - 6x) = 6y - 3x^2$	Both sides have a y' term: collect them on the left.
$y' = \frac{2y - x^2}{y^2 - 2x}$	Divide, and cancel the common factor 3.
$y' = \frac{6 - 9}{9 - 6} = -1$	Substitute.
Slope -1 ; tangent $y = -x + 6$	$y - 3 = -(x - 3)$. The curve is symmetric in x and y , and $(3, 3)$ is on the line of symmetry, so a slope of -1 is expected.

Problem 7

Find the points on $x^2 - xy + y^2 = 3$ where the tangent is horizontal, and where it is vertical.

$2x - (y + x y') + 2y y' = 0$	Product rule on xy , with its minus sign.
$y' = \frac{y - 2x}{2y - x}$	$y'(2y - x) = y - 2x$.

Horizontal: $y = 2x$, so $x^2 - 2x^2 + 4x^2 = 3x^2 = 3$ and $x = \pm 1$.	Numerator 0; substitute $y = 2x$ into the curve.
Vertical: $x = 2y$, so $4y^2 - 2y^2 + y^2 = 3y^2 = 3$ and $y = \pm 1$.	Denominator 0; substitute $x = 2y$.
Horizontal at $(1, 2)$ and $(-1, -2)$; vertical at $(2, 1)$ and $(-2, -1)$	At each point the other part of the fraction is not 0; at $(1, 2)$ the denominator is 3, at $(2, 1)$ the numerator is -3 .

Problem 8

For $x^2 + y^2 = 25$, find y'' in terms of y alone, and its value at $(3, 4)$.

$y' = -\frac{x}{y}$	Problem 1.
$y'' = -\frac{1 \cdot y - x y'}{y^2}$	Quotient rule; y is still a function of x , so its derivative is y' , not 0.
$y'' = -\frac{y + x^2/y}{y^2} = -\frac{y^2 + x^2}{y^3}$	Substitute $y' = -\frac{x}{y}$ and multiply top and bottom by y .
$y'' = -\frac{25}{y^3}$	Use the curve: $x^2 + y^2 = 25$.
$y'' = -\frac{25}{y^3}$; at $(3, 4)$, $-\frac{25}{64} \approx -0.391$	Negative on the upper half, where the circle is concave down.

Problem 9

Use $\sin y = x$ to find the derivative of $y = \arcsin x$, and its value at $x = \frac{1}{2}$.

$\cos y \ y' = 1$	Differentiate $\sin y = x$.
$y' = \frac{1}{\cos y}$	Divide.

$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$	$\arcsin$ takes values in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, where $\cos y \geq 0$, so the positive root is the right one.
$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1 - x^2}}$	For $-1 < x < 1$.
At $x = \frac{1}{2}$: $\frac{1}{\sqrt{3/4}} = \frac{2}{\sqrt{3}} \approx 1.155$	The same steps on $\tan y = x$ give $\frac{1}{1+x^2}$ for $\arctan$.

Problem 10

Use logarithmic differentiation to find y' for $y = \frac{(x+1)^2(x+2)^3}{(x+3)^4}$, and its value at $x = 0$.

$\ln y = 2 \ln(x+1) + 3 \ln(x+2) - 4 \ln(x+3)$	Take logs (for $x > -1$, where every factor is positive): products become sums and powers become coefficients.
$\frac{y'}{y} = \frac{2}{x+1} + \frac{3}{x+2} - \frac{4}{x+3}$	Differentiate; the left side is a chain rule on $\ln y$.
$y' = y \left(\frac{2}{x+1} + \frac{3}{x+2} - \frac{4}{x+3} \right)$	Multiply by y , then write y out in full.
At $x = 0$: $y = \frac{1 \cdot 8}{81} = \frac{8}{81}$ and $\frac{y'}{y} = 2 + \frac{3}{2} - \frac{4}{3} = \frac{13}{6}$.	Substitute.
$y' = \frac{8}{81} \cdot \frac{13}{6} = \frac{52}{243} \approx 0.214$	The quotient rule gets the same answer through a much longer numerator.

Where this goes wrong

1. Dropping the y' on a y term

Differentiating y^2 gives $2y$ in every exercise where y is the variable.

$x^2 + 3xy + y^2 = 11$ at $(1, 2)$	Right so far: Problem 3.
------------------------------------	--------------------------

$$\text{“} \frac{d}{dx} y^2 = 2y \text{.”}$$

The habit that causes the mistake: differentiating y^2 as if y were the variable of differentiation.

$$2x + 3y + 3x y' + 2y = 0, \text{ so } y' = -\frac{2x + 5y}{3x} = -4 \text{ } \times$$

y depends on x , so $\frac{d}{dx} y^2 = 2y y'$. With the factor, $y'(3x + 2y) = -(2x + 3y)$ and $y' = -\frac{8}{7}$. Every term with a y in it must produce a y' .

2. Skipping the product rule on xy

$3xy$ looks like a single term, so it gets a single derivative.

$$x^2 + 3xy + y^2 = 11 \text{ at } (1, 2)$$

Right so far: Problem 3.

$$\text{“} \frac{d}{dx} (3xy) = 3x y' \text{.”}$$

The habit that causes the mistake: treating $3x$ as a constant coefficient of y .

$$2x + 3x y' + 2y y' = 0, \text{ so } y' = -\frac{2x}{3x + 2y} = -\frac{2}{7} \text{ } \times$$

x changes too, so $3xy$ is a product: $\frac{d}{dx} (3xy) = 3y + 3x y'$. The missing $3y$ changes the numerator from $2x + 3y = 8$ to $2x = 2$, and the answer is $-\frac{8}{7}$, four times as steep.

3. Differentiating the constant side

The right side is π , and π looks like an angle, so it is treated like a variable.

$$x \sin y + y = \pi \text{ at } \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Right so far: Problem 5.

$$\text{“} \frac{d}{dx} \pi = 1 \text{.”}$$

The habit that causes the mistake: differentiating every symbol that is not a digit.

$$\sin y + x \cos y y' + y' = 1, \text{ so } 1 + y' = 1 \text{ and } y' = 0 \text{ } \times$$

π is a number, and the derivative of a number is 0. With 0 on the right, $1 + y' = 0$ and $y' = -1$. A horizontal tangent here would be a coincidence; check the answer against the curve instead.

4. Leaving a y' term on the other side

After differentiating, the left side has a y' and is solved for it, the way an explicit derivative would be.

$$3x^2 + 3y^2 y' = 6y + 6x y' \text{ at } (3, 3)$$

Right so far: Problem 6, step 2.

$$“y' = \frac{6y - 3x^2}{3y^2}.”$$

The habit that causes the mistake:
isolating the y' on the left and treating
the right side as known.

$$y' = \frac{18 - 27}{27} = -\frac{1}{3} \text{ ✗}$$

The right side still contains $6x y'$, so this
is not a solution for y' : y' appears on
both sides. Collect the terms first,
 $y'(3y^2 - 6x) = 6y - 3x^2$, and the slope is
 $\frac{-9}{9} = -1$.

5. Treating y as a constant in the second derivative

In $-\frac{x}{y}$ the y sits in the denominator like a constant, so only the x gets differentiated.

$$y' = -\frac{x}{y} \text{ on } x^2 + y^2 = 25, \text{ at } (3, 4)$$

Right so far: Problem 8, step 1.

$$“y'' = -\frac{1}{y}.”$$

The habit that causes the mistake:
forgetting that y is a function of x once
the first derivative is done.

$$y'' = -\frac{1}{4} \text{ at } (3, 4) \text{ ✗}$$

The quotient rule needs the derivative of
the denominator, which is $y' = -\frac{3}{4}$, not
0: $y'' = -\frac{y - x y'}{y^2} = -\frac{4 + 9/4}{16} = -\frac{25}{64}$. Then
substitute y' and use the curve to
simplify, as in Problem 8.