

The fundamental theorem of calculus: derivatives of integrals and definite integrals

Ten problems on both parts of the fundamental theorem: an accumulation function checked against its closed form, a chain-rule upper limit, a variable lower limit, two moving limits and the maximum they give, critical points of an integral, definite integrals with an absolute value, the derivative of a Gaussian and an exponential CDF, and a limit of an integral by L'Hôpital's rule, with worked solutions and the mistakes of a missing chain-rule factor, a lost minus sign and confusing F with f .

BEFORE YOU START

The fundamental theorem is the link between the two halves of calculus: the derivative of an accumulated area is the height of the curve, and a definite integral is a difference of antiderivatives. In machine learning it is the reason a probability density is the derivative of its cumulative distribution function, so the density of any transformed variable comes from the chain rule on a CDF. The work goes wrong in a few places: a chain-rule factor left off a limit like x^3 , the minus sign of a variable lower limit dropped, the value of the integrand at a constant lower limit subtracted, an absolute value integrated as if it were not there, and the accumulation function F confused with its integrand f . These ten problems cover each of them.

- **Part 1.** If f is continuous and $F(x) = \int_a^x f(t) dt$, then $F'(x) = f(x)$. The constant a does not appear in the answer.
- **Moving limits.** By the chain rule, $\frac{d}{dx} \int_{g(x)}^{h(x)} f(t) dt = f(h(x)) h'(x) - f(g(x)) g'(x)$. A constant limit contributes 0; a variable lower limit contributes a minus sign.
- **Part 2.** If f is continuous on $[a, b]$ and $F' = f$, then $\int_a^b f(x) dx = F(b) - F(a)$. Any antiderivative works, so no $+C$ is needed.
- $F(x) = \int_a^x f(t) dt$ is a function of x , not of t : t is a dummy variable. F is increasing where $f > 0$, so its critical points are the zeros of f .
- A **cumulative distribution function** is $F(x) = \int_{-\infty}^x p(t) dt$ for a density p , so $F' = p$. For the standard normal, $\Phi(z) = \int_{-\infty}^z \varphi(t) dt$ with $\varphi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$.
- $|x - c|$ is $c - x$ for $x < c$ and $x - c$ for $x > c$: split the integral at c .

Builds on: Differentiation rules: chain, product, quotient

Problems

1. Let $F(x) = \int_1^x (t^3 - 2t) dt$. Find $F'(x)$ by Part 1, then find $F(x)$ in closed form and check.

difficulty ·

2. Find $\frac{d}{dx} \int_1^{x^3} \ln t dt$ for $x > 0$, and its value at $x = 2$.

difficulty ··

3. Find $\frac{d}{dx} \int_x^5 \sqrt{1+t^4} dt$, and its value at $x = 1$.

difficulty ··

4. Let $G(x) = \int_x^{2x} e^{-t} dt$ for $x \geq 0$. Find $G'(x)$, and the x at which G is largest.

difficulty ...

5. Let $F(x) = \int_0^x (t^2 - 4t + 3) dt$. Find the local maximum and minimum values of F .

difficulty ··

6. Evaluate $\int_1^4 \left(2x + \frac{1}{\sqrt{x}}\right) dx$.

difficulty ·

7. Evaluate $\int_0^3 |x - 1| dx$.

difficulty ··

8. Φ is the standard normal CDF. Find $\frac{d}{dx} \Phi(2x - 1)$, and its value at $x = \frac{1}{2}$.

difficulty ··

9. The waiting time T has density $p(t) = 2e^{-2t}$ for $t \geq 0$. Find its CDF $F(x) = P(T \leq x)$, check that $F' = p$, and find the median.

difficulty ...

10. Find $\lim_{x \rightarrow 0} \frac{1}{x^3} \int_0^x \sin(t^2) dt$.

difficulty ...

Answers

1. $F'(x) = x^3 - 2x$, and $F(x) = \frac{x^4}{4} - x^2 + \frac{3}{4}$ has that derivative
2. $9x^2 \ln x$; at $x = 2$ it is $36 \ln 2 \approx 24.95$
3. $-\sqrt{1+x^4}$; at $x = 1$ it is $-\sqrt{2} \approx -1.414$
4. $G'(x) = 2e^{-2x} - e^{-x}$; G is largest at $x = \ln 2 \approx 0.693$, where $G = \frac{1}{4}$
5. Local maximum $\frac{4}{3}$ at $x = 1$; local minimum 0 at $x = 3$
6. 17
7. $\frac{5}{2}$
8. $2\varphi(2x - 1)$; at $x = \frac{1}{2}$ it is $\sqrt{2/\pi} \approx 0.798$
9. $F(x) = 1 - e^{-2x}$; median $\frac{\ln 2}{2} \approx 0.347$
10. $\frac{1}{3}$

Worked solutions

Problem 1

Let $F(x) = \int_1^x (t^3 - 2t) dt$. Find $F'(x)$ by Part 1, then find $F(x)$ in closed form and check.

$$F'(x) = x^3 - 2x$$

Part 1: replace t with the upper limit.

$$F(x) = \left[\frac{t^4}{4} - t^2 \right]_1^x = \frac{x^4}{4} - x^2 - \left(\frac{1}{4} - 1 \right)$$

Part 2 with the antiderivative $\frac{t^4}{4} - t^2$.

$$F(x) = \frac{x^4}{4} - x^2 + \frac{3}{4}$$

The lower limit only adds the constant $\frac{3}{4}$.

$$F'(x) = x^3 - 2x, \text{ and } F(x) = \frac{x^4}{4} - x^2 + \frac{3}{4}$$

has that derivative

The constant $\frac{3}{4}$ differentiates to 0, which is why the lower limit never appears in F' . Also $F(1) = 0$, as it must.

Problem 2

Find $\frac{d}{dx} \int_1^{x^3} \ln t dt$ for $x > 0$, and its value at $x = 2$.

$$\ln(x^3) \cdot \frac{d}{dx} x^3$$

Put the upper limit into the integrand, then multiply by its derivative.

$$= 3 \ln x \cdot 3x^2$$

$$\ln(x^3) = 3 \ln x \text{ and } (x^3)' = 3x^2.$$

Check: $\int_1^{x^3} \ln t dt = [t \ln t - t]_1^{x^3} =$
 $3x^3 \ln x - x^3 + 1$, whose derivative is
 $9x^2 \ln x + 3x^2 - 3x^2$

The long way agrees.

$$9x^2 \ln x; \text{ at } x = 2 \text{ it is } 36 \ln 2 \approx 24.95$$

Without the chain-rule factor the answer would be $3 \ln x$, off by a factor of $3x^2$.

Problem 3

Find $\frac{d}{dx} \int_x^5 \sqrt{1+t^4} dt$, and its value at $x = 1$.

$$\int_x^5 \sqrt{1+t^4} dt = - \int_5^x \sqrt{1+t^4} dt$$

Swapping the limits changes the sign, and puts x in the upper limit.

$$\frac{d}{dx} \left(- \int_5^x \sqrt{1+t^4} dt \right) = -\sqrt{1+x^4}$$

Part 1.

$$-\sqrt{1+x^4}; \text{ at } x = 1 \text{ it is } -\sqrt{2} \approx -1.414$$

Moving the lower limit right removes area from a positive integrand, so the derivative must be negative.

Problem 4

Let $G(x) = \int_x^{2x} e^{-t} dt$ for $x \geq 0$. Find $G'(x)$, and the x at which G is largest.

$$G'(x) = e^{-2x} \cdot 2 - e^{-x} \cdot 1$$

Upper limit $2x$ with derivative 2, minus the lower limit x with derivative 1.

$$G'(x) = 0 \iff 2e^{-2x} = e^{-x} \iff e^{-x} = \frac{1}{2}$$

Divide by $e^{-x} > 0$.

$$x = \ln 2$$

$G' > 0$ for $x < \ln 2$ and $G' < 0$ after, so this is the maximum.

$$G(\ln 2) = e^{-\ln 2} - e^{-2 \ln 2} = \frac{1}{2} - \frac{1}{4}$$

$G(x) = e^{-x} - e^{-2x}$ by Part 2.

$$G'(x) = 2e^{-2x} - e^{-x}; G \text{ is largest at } x = \ln 2 \approx 0.693, \text{ where } G = \frac{1}{4}$$

$G(0) = 0$ and $G \rightarrow 0$ as $x \rightarrow \infty$: the interval $[x, 2x]$ starts empty and ends where e^{-t} is tiny.

Problem 5

Let $F(x) = \int_0^x (t^2 - 4t + 3) dt$. Find the local maximum and minimum values of F .

$$F'(x) = x^2 - 4x + 3 = (x-1)(x-3)$$

Part 1. The critical points of F are the zeros of its integrand.

$F''(x) = 2x - 4$, so $F''(1) = -2$ and $F''(3) = 2$	Local maximum at 1, local minimum at 3.
$F(x) = \frac{x^3}{3} - 2x^2 + 3x$	Part 2; the lower limit 0 adds nothing.
$F(1) = \frac{1}{3} - 2 + 3 = \frac{4}{3}$ and $F(3) = 9 - 18 + 9 = 0$	The values of F , not of the integrand.
Local maximum $\frac{4}{3}$ at $x = 1$; local minimum 0 at $x = 3$	F climbs while the integrand is positive, on $(0, 1)$, and loses exactly that area again on $(1, 3)$, where the integrand is negative.

Problem 6

Evaluate $\int_1^4 \left(2x + \frac{1}{\sqrt{x}}\right) dx$.	
An antiderivative is $x^2 + 2\sqrt{x}$	$\int x^{-1/2} dx = \frac{x^{1/2}}{1/2} = 2\sqrt{x}$.
$(16 + 2 \cdot 2) - (1 + 2 \cdot 1) = 20 - 3$	$F(4) - F(1)$. The integrand is continuous on $[1, 4]$.
17	Term by term: $\int_1^4 2x dx = 15$ and $\int_1^4 x^{-1/2} dx = 2$.

Problem 7

Evaluate $\int_0^3 x - 1 dx$.	
$ x - 1 = 1 - x$ on $[0, 1]$ and $x - 1$ on $[1, 3]$	Split where the inside changes sign.
$\int_0^1 (1 - x) dx = \left[x - \frac{x^2}{2}\right]_0^1 = \frac{1}{2}$	The first piece.

$\int_1^3 (x-1) dx = \left[\frac{x^2}{2} - x \right]_1^3 = \frac{3}{2} - \left(-\frac{1}{2} \right) = 2$	The second piece.
$\frac{5}{2}$	Two triangles: one with legs 1 and 1, one with legs 2 and 2.

Problem 8

Φ is the standard normal CDF. Find $\frac{d}{dx}\Phi(2x-1)$, and its value at $x = \frac{1}{2}$.	
$\Phi(2x-1) = \int_{-\infty}^{2x-1} \varphi(t) dt$	A CDF is an integral with a variable upper limit.
$\frac{d}{dx}\Phi(2x-1) = \varphi(2x-1) \cdot 2$	Part 1 with the chain rule; the lower limit $-\infty$ does not move.
At $x = \frac{1}{2}$: $2\varphi(0) = \frac{2}{\sqrt{2\pi}}$	$\varphi(0) = \frac{1}{\sqrt{2\pi}}$.
$2\varphi(2x-1)$; at $x = \frac{1}{2}$ it is $\sqrt{2/\pi} \approx 0.798$	This is the density of $X = \frac{Z+1}{2}$: $P(X \leq x) = \Phi(2x-1)$, and differentiating a CDF gives the density. It is the normal density with mean $\frac{1}{2}$ and standard deviation $\frac{1}{2}$.

Problem 9

The waiting time T has density $p(t) = 2e^{-2t}$ for $t \geq 0$. Find its CDF $F(x) = P(T \leq x)$, check that $F' = p$, and find the median.	
$F(x) = \int_0^x 2e^{-2t} dt = \left[-e^{-2t} \right]_0^x$	The density is 0 for $t < 0$, so the integral starts at 0.
$F(x) = 1 - e^{-2x} \text{ for } x \geq 0$	$-e^{-2x} - (-1)$. $F(0) = 0$ and $F \rightarrow 1$.
$F'(x) = 2e^{-2x} = p(x)$	Part 1, confirmed by differentiating the closed form.

$$1 - e^{-2m} = \frac{1}{2} \iff e^{-2m} = \frac{1}{2} \iff m = \frac{\ln 2}{2}$$

The median is where the CDF reaches $\frac{1}{2}$.

$$F(x) = 1 - e^{-2x}; \text{ median } \frac{\ln 2}{2} \approx 0.347$$

The mean is $\frac{1}{2}$, larger than the median: the long right tail pulls it up.

Problem 10

$$\text{Find } \lim_{x \rightarrow 0} \frac{1}{x^3} \int_0^x \sin(t^2) dt.$$

As $x \rightarrow 0$ both $\int_0^x \sin(t^2) dt$ and x^3 go to 0

The form is $\frac{0}{0}$, so L'Hôpital's rule applies.

$$\lim_{x \rightarrow 0} \frac{\sin(x^2)}{3x^2}$$

Differentiate the top by Part 1 and the bottom by the power rule.

$$\frac{\sin(x^2)}{x^2} \rightarrow 1$$

With $u = x^2 \rightarrow 0$, $\frac{\sin u}{u} \rightarrow 1$.

$$\frac{1}{3}$$

Near 0, $\sin(t^2) \approx t^2$, and $\int_0^x t^2 dt = \frac{x^3}{3}$: the same answer without the rule.

Where this goes wrong

1. Leaving out the chain-rule factor

Part 1 is learned as "put the upper limit in", which is the whole rule only when the limit is x .

$$\frac{d}{dx} \int_1^{x^3} \ln t dt, x > 0$$

Right so far: Problem 2.

"Replace t with the upper limit."

The habit that causes the mistake: the limit is x^3 , not x , so it moves $3x^2$ times as fast.

$\ln(x^3) = 3 \ln x$, so at $x = 2$ the derivative is $3 \ln 2 \approx 2.079$ ✗

The edge of the area moves at the rate $(x^3)' = 3x^2$, and the derivative is $9x^2 \ln x$, which is $36 \ln 2 \approx 24.95$ at $x = 2$, twelve times as much. Differentiating the closed form $3x^3 \ln x - x^3 + 1$ shows the factor.

2. Dropping the sign of a variable lower limit

In most problems x is the upper limit, so the minus sign is easy to forget.

$$\frac{d}{dx} \int_x^5 \sqrt{1+t^4} dt$$

Right so far: Problem 3.

“Put x into the integrand.”

The habit that causes the mistake: Part 1 applied without first moving x to the upper limit.

The derivative is $\sqrt{1+x^4}$, so $\sqrt{2} \approx 1.414$ at $x = 1$ ✗

$\int_x^5 = -\int_5^x$, so the derivative is $-\sqrt{1+x^4}$, which is -1.414 at $x = 1$. As x increases, the interval $[x, 5]$ gets shorter and the integrand is positive, so the integral is decreasing.

3. Subtracting the integrand at the lower limit

Part 2 is "top minus bottom", and the pattern carries over to Part 1, where it does not belong.

$$F(x) = \int_1^x (t^3 - 2t) dt$$

Right so far: Problem 1.

“The derivative is f at the top minus f at the bottom.”

The habit that causes the mistake: Part 2's subtraction copied into Part 1.

$$F'(x) = (x^3 - 2x) - (1 - 2) = x^3 - 2x + 1$$

✗

The lower limit is a constant, and it contributes a constant, $\frac{3}{4}$, to F , which differentiates to 0. $F'(x) = x^3 - 2x$; at $x = 2$ that is 4, not 5. Only a lower limit that moves with x contributes to the derivative.

4. Integrating an absolute value without splitting

The bars look like decoration once the integral is set up.

$$\int_0^3 |x - 1| dx$$

Right so far: Problem 7.

“For most of the interval $x - 1 \geq 0$, so use $x - 1$.”

The habit that causes the mistake: one formula used on both sides of $x = 1$.

$$\int_0^3 (x - 1) dx = \frac{9}{2} - 3 = \frac{3}{2} \quad \times$$

On $[0, 1]$, $x - 1$ is negative, so the triangle there, of area $\frac{1}{2}$, is subtracted instead of added. Split at $x = 1$: $\frac{1}{2} + 2 = \frac{5}{2}$. The integral of a function that is never negative is a total area, and here $\frac{3}{2} = 2 - \frac{1}{2}$ is a net one.

5. Reporting the integrand's value instead of F's

F and its integrand share the same critical points, which makes it easy to evaluate the wrong one.

$$F(x) = \int_0^x (t^2 - 4t + 3) dt, \text{ local maximum at } x = 1$$

Right so far: Problem 5, step 2.

“Plug $x = 1$ into the function.”

The habit that causes the mistake: the function on the page is the integrand.

The local maximum value is $1 - 4 + 3 = 0$
 $\times$

0 is the integrand at $x = 1$: it is $F'(1)$, which is 0 at every critical point. The value of F is the area accumulated up to 1, $F(1) = \frac{1}{3} - 2 + 3 = \frac{4}{3}$. The local minimum value is $F(3) = 0$, which happens to be 0 for a different reason.