

Evaluating limits: factoring, conjugates and limits at infinity

Ten limits evaluated exactly: o/o forms cleared by factoring, by a difference of cubes and by the conjugate, the difference quotient behind every derivative, $\sin 3x / x$, one-sided limits that disagree, and limits at $\pm\infty$ including $\sqrt{x^2} = |x|$ and $\infty - \infty$, with worked solutions and the mistakes that read o/o as an answer.

BEFORE YOU START

A limit asks what value $f(x)$ approaches as x approaches a , not what $f(a)$ is, and every derivative is one of them. When substituting a gives a number, that number is the limit; when it gives $\frac{0}{0}$ or $\frac{\infty}{\infty}$, the work is to rewrite f until substitution does give a number. These ten problems run from factoring and the conjugate to the difference quotient, one-sided limits, and limits at infinity, where the sign of $\sqrt{x^2}$ and the form $\infty - \infty$ catch people out.

- $\lim_{x \rightarrow a} f(x) = L$ means $f(x)$ gets as close to L as you like for every x close enough to a , with $x \neq a$. The value $f(a)$, or whether $f(a)$ is defined at all, plays no part.
- If f is built from polynomials, roots, exponentials, logs and trig functions by $+$, $-$, $\times$, $\div$ and composition, and $f(a)$ is defined, then $\lim_{x \rightarrow a} f(x) = f(a)$: substitute.
- $\frac{0}{0}$ is not a value. It says the limit depends on how fast the top and bottom go to 0, and the expression has to be rewritten before you substitute. Two expressions that agree for every $x \neq a$ have the same limit at a , which is what makes canceling a common factor $(x - a)$ legal.
- One-sided limits: $x \rightarrow a^-$ approaches from below, $x \rightarrow a^+$ from above. The two-sided limit exists exactly when both one-sided limits exist and are equal.
- Standard limit: $\lim_{t \rightarrow 0} \frac{\sin t}{t} = 1$, with t in radians.
- At $\pm\infty$, divide the top and bottom by the highest power of x in the denominator, and use $\frac{1}{x^n} \rightarrow 0$ for $n > 0$. For a root, $\sqrt{x^2} = |x|$, which is $-x$ when $x < 0$.

Problems

1. Find $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$.

difficulty ·

2. Find $\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x^2 - 4}$.

difficulty ·

3. Find $\lim_{x \rightarrow 0} \frac{\sqrt{x+4} - 2}{x}$.

difficulty ..

4. Find $\lim_{x \rightarrow 1} \frac{x^3 - 1}{x^2 - 1}$.

difficulty ..

5. Find $\lim_{h \rightarrow 0} \frac{(2+h)^2 - 4}{h}$, and say which derivative it is.

difficulty ..

6. Find $\lim_{x \rightarrow \infty} \frac{3x^2 - x + 5}{2x^2 + 7}$.

difficulty ..

7. Find $\lim_{x \rightarrow -\infty} \frac{\sqrt{4x^2 + 1}}{x + 3}$.

difficulty ...

8. Find $\lim_{x \rightarrow 2^-} \frac{|x-2|}{x-2}$ and $\lim_{x \rightarrow 2^+} \frac{|x-2|}{x-2}$. Does $\lim_{x \rightarrow 2} \frac{|x-2|}{x-2}$ exist?

difficulty ..

9. Find $\lim_{x \rightarrow 0} \frac{\sin 3x}{x}$.

difficulty ...

10. Find $\lim_{x \rightarrow \infty} \left(\sqrt{x^2 + 6x} - x \right)$.

difficulty ...

Answers

1. 6
2. $\frac{5}{4}$
3. $\frac{1}{4}$
4. $\frac{3}{2}$
5. 4; it is $f'(2)$ for $f(x) = x^2$
6. $\frac{3}{2}$
7. -2
8. -1 from the left, 1 from the right; the two-sided limit does not exist
9. 3
10. 3

Worked solutions

Problem 1

Find $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$.

Substituting $x = 3$ gives $\frac{9-9}{3-3} = \frac{0}{0}$.

Not an answer: the expression has to be rewritten first.

$$\frac{x^2 - 9}{x - 3} = \frac{(x - 3)(x + 3)}{x - 3} = x + 3 \text{ for } x \neq 3.$$

Difference of squares. The factor $x - 3$ is the one that made both the top and the bottom 0, and canceling it is legal because the limit never uses $x = 3$ itself.

$$\lim_{x \rightarrow 3} (x + 3) = 6.$$

$x + 3$ is a polynomial, so substitute.

6

Steps 2 and 3.

Problem 2

Find $\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x^2 - 4}$.

Substituting $x = 2$ gives $\frac{4+2-6}{4-4} = \frac{0}{0}$.

Both polynomials are 0 at 2, so both have the factor $x - 2$.

$$x^2 + x - 6 = (x - 2)(x + 3) \text{ and } x^2 - 4 = (x - 2)(x + 2).$$

Factor both. In the first, -2 and 3 multiply to -6 and add to 1 .

$$\frac{x^2 + x - 6}{x^2 - 4} = \frac{x + 3}{x + 2} \text{ for } x \neq 2.$$

Cancel the common factor $x - 2$.

$\frac{5}{4}$

Substitute $x = 2$ into $\frac{x+3}{x+2}$, which is defined there: $\frac{5}{4}$.

Problem 3

Find $\lim_{x \rightarrow 0} \frac{\sqrt{x+4} - 2}{x}$.

Substituting $x = 0$ gives $\frac{2-2}{0} = \frac{0}{0}$.

There is no polynomial to factor, so use the conjugate of the root expression.

$$\frac{\sqrt{x+4} - 2}{x} \cdot \frac{\sqrt{x+4} + 2}{\sqrt{x+4} + 2}$$

Multiplying by $\frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}$, which is 1, changes the form and not the value. The top and the bottom both get the factor.

$$(\sqrt{x+4} - 2)(\sqrt{x+4} + 2) = (x+4) - 4 = x.$$

Difference of squares, $(p-q)(p+q) = p^2 - q^2$: the root squares away.

$$\frac{x}{x(\sqrt{x+4} + 2)} = \frac{1}{\sqrt{x+4} + 2} \text{ for } x \neq 0.$$

Step 3 on top, the bottom left in factored form so the x can cancel.

$$\frac{1}{4}$$

Substitute $x = 0$: $\frac{1}{\sqrt{4}+2} = \frac{1}{4}$.

Problem 4

Find $\lim_{x \rightarrow 1} \frac{x^3 - 1}{x^2 - 1}$.

Substituting $x = 1$ gives $\frac{0}{0}$.

Both are 0 at $x = 1$, so both have the factor $x - 1$.

$$x^3 - 1 = (x - 1)(x^2 + x + 1) \text{ and } x^2 - 1 = (x - 1)(x + 1).$$

Difference of cubes, $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$, and difference of squares.

$$\frac{x^3 - 1}{x^2 - 1} = \frac{x^2 + x + 1}{x + 1} \text{ for } x \neq 1.$$

Cancel $x - 1$.

$$\frac{3}{2}$$

Substitute $x = 1$: $\frac{1+1+1}{1+1} = \frac{3}{2}$.

Problem 5

Find $\lim_{h \rightarrow 0} \frac{(2+h)^2 - 4}{h}$, and say which derivative it is.

$$(2+h)^2 - 4 = 4 + 4h + h^2 - 4 = 4h + h^2.$$

Expand the square, middle term included. Substituting $h = 0$ in the original gives $\frac{0}{0}$, so it has to be rewritten.

$$\frac{4h + h^2}{h} = 4 + h \text{ for } h \neq 0.$$

Every term on top has a factor h , so it cancels with the bottom.

$$\lim_{h \rightarrow 0} (4 + h) = 4.$$

Substitute $h = 0$.

$$4; \text{ it is } f'(2) \text{ for } f(x) = x^2$$

The derivative is defined as $f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$, and here $f(a+h) = (2+h)^2$, $f(a) = 4$. It agrees with $(x^2)' = 2x$ at $x = 2$.

Problem 6

Find $\lim_{x \rightarrow \infty} \frac{3x^2 - x + 5}{2x^2 + 7}$.

Substituting is not possible, and the top and bottom both grow without bound: the form is $\frac{\infty}{\infty}$.

Rewrite so that every term has a limit.

$$\frac{3x^2 - x + 5}{2x^2 + 7} = \frac{3 - \frac{1}{x} + \frac{5}{x^2}}{2 + \frac{7}{x^2}}$$

Divide every term, top and bottom, by x^2 , the highest power in the denominator. This is multiplying by $\frac{1/x^2}{1/x^2} = 1$.

$$\frac{1}{x} \rightarrow 0, \frac{5}{x^2} \rightarrow 0 \text{ and } \frac{7}{x^2} \rightarrow 0 \text{ as } x \rightarrow \infty.$$

A fixed number divided by an ever larger one.

$$\frac{3}{2}$$

The limit of the top is 3 and of the bottom is 2. When the top and bottom have the same degree, the limit is the ratio of the leading coefficients.

Problem 7

Find $\lim_{x \rightarrow -\infty} \frac{\sqrt{4x^2 + 1}}{x + 3}$.

$$\sqrt{4x^2 + 1} = \sqrt{x^2} \sqrt{4 + \frac{1}{x^2}} = |x| \sqrt{4 + \frac{1}{x^2}}.$$

Factor x^2 out from under the root. $\sqrt{x^2}$ is $|x|$, not x : the root is never negative.

$$|x| = -x \text{ for } x < 0.$$

$x \rightarrow -\infty$, so from some point on x is negative.

$$\frac{\sqrt{4x^2 + 1}}{x + 3} = \frac{-x \sqrt{4 + \frac{1}{x^2}}}{x(1 + \frac{3}{x})} = \frac{-\sqrt{4 + \frac{1}{x^2}}}{1 + \frac{3}{x}}$$

Steps 1 and 2 on top, x factored out of the bottom, then cancel x .

$$-2$$

$\frac{1}{x^2} \rightarrow 0$ and $\frac{3}{x} \rightarrow 0$, so the limit is $\frac{-\sqrt{4}}{1} = -2$. Sanity check: the top is positive and the bottom is negative for large negative x , so the limit has to be negative.

Problem 8

Find $\lim_{x \rightarrow 2^-} \frac{|x - 2|}{x - 2}$ and $\lim_{x \rightarrow 2^+} \frac{|x - 2|}{x - 2}$. Does $\lim_{x \rightarrow 2} \frac{|x - 2|}{x - 2}$ exist?

For $x < 2$, $|x - 2| = -(x - 2)$, so

$$\frac{|x - 2|}{x - 2} = -1.$$

Below 2, $x - 2$ is negative and the absolute value flips its sign.

$$\lim_{x \rightarrow 2^-} \frac{|x - 2|}{x - 2} = -1.$$

The expression is the constant -1 for every x just below 2.

For $x > 2$, $|x - 2| = x - 2$, so the expression is 1 and $\lim_{x \rightarrow 2^+} = 1$.

Above 2, $x - 2$ is positive and the absolute value changes nothing.

**-1 from the left, 1 from the right;
the two-sided limit does not exist**

The one-sided limits exist but differ, so there is no single value that $f(x)$ approaches from both sides.

Problem 9

Find $\lim_{x \rightarrow 0} \frac{\sin 3x}{x}$.

Substituting $x = 0$ gives $\frac{\sin 0}{0} = \frac{0}{0}$.

The standard limit $\frac{\sin t}{t} \rightarrow 1$ applies only when the angle and the denominator are the same quantity.

$$\frac{\sin 3x}{x} = 3 \cdot \frac{\sin 3x}{3x}$$

Multiply the top and the bottom by 3, so the denominator matches the angle.

$t = 3x \rightarrow 0$ as $x \rightarrow 0$, so $\frac{\sin 3x}{3x} = \frac{\sin t}{t} \rightarrow 1$.

The substitution $t = 3x$ turns the expression into the standard limit.

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$3 \cdot 1$. The same argument gives $\lim_{x \rightarrow 0} \frac{\sin kx}{x} = k$ for any constant k .

Problem 10

Find $\lim_{x \rightarrow \infty} (\sqrt{x^2 + 6x} - x)$.

Both $\sqrt{x^2 + 6x}$ and x go to ∞ : the form is $\infty - \infty$.

Like $\frac{0}{0}$, that is not a value. Multiply by the conjugate to turn the difference into a quotient.

$$\begin{aligned} & (\sqrt{x^2 + 6x} - x) \frac{\sqrt{x^2 + 6x} + x}{\sqrt{x^2 + 6x} + x} = \\ & \frac{(x^2 + 6x) - x^2}{\sqrt{x^2 + 6x} + x} \end{aligned}$$

Difference of squares on top: the x^2 terms cancel.

$$\begin{aligned} & \frac{6x}{\sqrt{x^2 + 6x} + x} = \frac{6x}{x\sqrt{1 + \frac{6}{x}} + x} = \\ & \frac{6}{\sqrt{1 + \frac{6}{x}} + 1} \end{aligned}$$

$\sqrt{x^2 + 6x} = |x|\sqrt{1 + \frac{6}{x}}$ and $|x| = x$ for $x > 0$; then cancel x .

3

$\frac{6}{x} \rightarrow 0$, so the limit is $\frac{6}{1+1} = 3$. The root is about $x + 3$ for large x , since $(x + 3)^2 = x^2 + 6x + 9$.

Where this goes wrong

1. Reading 0/0 as the answer

Substitution is the first thing to try, and when it produces $\frac{0}{0}$ it looks like a result.

$$\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} : \text{substituting gives } \frac{0}{0}$$

Right so far: Problem 1, step 1.

“Division by zero is undefined, so the limit is undefined.”

The habit that causes the mistake: treating the limit as the value at $x = 3$.

$$\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} \text{ does not exist } \times$$

The function is undefined at 3, but the limit never uses $x = 3$. For every other x it equals $x + 3$, which approaches 6; at $x = 3.001$ the expression is 6.001. $\frac{0}{0}$ is a signal to rewrite, never a final answer, and neither is "0" or "1".

2. Canceling terms instead of factors

The x^2 on top and the x^2 on the bottom look like they should cancel.

$$\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x^2 - 4}$$

Right so far: Problem 2.

“Cancel the x^2 from the top and the bottom.”

The habit that causes the mistake: canceling a term that is added, not multiplied.

$$\frac{x^2 + x - 6}{x^2 - 4} = \frac{x - 6}{-4}, \text{ so the limit is } \frac{-4}{-4} = 1$$

$\times$

Only a factor of the whole top and the whole bottom cancels. At $x = 3$ the left side is $\frac{6}{5}$ and the right side is $\frac{3}{4}$, so the two are not equal. Factor first: the common factor is $x - 2$, and the limit is $\frac{5}{4}$ (Problem 2).

3. Multiplying only the top by the conjugate

The conjugate is there to remove the root from the top, so the bottom gets left alone.

$$\lim_{x \rightarrow 0} \frac{\sqrt{x+4} - 2}{x}, \text{ using the conjugate}$$

$$\sqrt{x+4} + 2$$

Right so far: Problem 3.

“($\sqrt{x+4} - 2$)($\sqrt{x+4} + 2$) = x , so the numerator becomes x .”

The numerator is right; the bottom has not been touched.

$$\frac{\sqrt{x+4} - 2}{x} = \frac{x}{x} = 1 \quad \text{✗}$$

Multiplying only the top by $\sqrt{x+4} + 2$ multiplies the whole expression by a number near 4, which changes it. The bottom must become $x(\sqrt{x+4} + 2)$; the answer is $\frac{1}{4}$, not 1.

4. Writing $\sqrt{x^2} = x$ when $x \rightarrow -\infty$

Pulling x out of a square root is a reflex, and for positive x it is right.

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{4x^2 + 1}}{x + 3}$$

Right so far: Problem 7.

$$\sqrt{4x^2 + 1} = x\sqrt{4 + 1/x^2}.$$

The habit that causes the mistake:
 $\sqrt{x^2} = x$, which holds only for $x \geq 0$.

$$\frac{x\sqrt{4 + 1/x^2}}{x(1 + 3/x)} \rightarrow 2 \quad \text{✗}$$

$\sqrt{x^2} = |x| = -x$ for negative x , so a minus sign is missing. At $x = -1000$ the expression is about $\frac{2000}{-997} \approx -2.006$: a positive root over a negative denominator. The limit is -2 .

5. Treating $\infty - \infty$ as 0

For large x , $\sqrt{x^2 + 6x}$ and x are both huge and nearly equal, so their difference looks negligible.

$$\lim_{x \rightarrow \infty} (\sqrt{x^2 + 6x} - x)$$

Right so far: Problem 10.

“For large x , $x^2 + 6x \approx x^2$, so
 $\sqrt{x^2 + 6x} \approx x$.”

True as a ratio, $\sqrt{x^2 + 6x}/x \rightarrow 1$, but a ratio near 1 says nothing about the difference.

$$\lim_{x \rightarrow \infty} (\sqrt{x^2 + 6x} - x) = \infty - \infty = 0 \text{ ✗}$$

At $x = 10^6$ the difference is $2.999995 \dots$, nowhere near 0. The $6x$ that was dropped is exactly what the difference measures. Use the conjugate (Problem 10): the limit is 3.

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