Conic sections: parabolas, ellipses and hyperbolas

Ten problems on conic sections: the focus and directrix of a parabola, the foci and eccentricity of an ellipse and the constant sum of distances, a hyperbola's foci, asymptotes and constant difference, completing the square to find the center, building equations from foci and asymptotes, classifying a conic with an xy term by its discriminant, rotating the axes to remove the xy term, a tangent line to an ellipse by implicit differentiation, a conic in polar form with its focus at the origin, and the area of an ellipse by integration, with worked solutions and the mistakes of using c² = a² + b² for an ellipse, taking the asymptote slope upside down, completing the square without the coefficient, reading a and b without the square roots and classifying by the signs of x² and y² alone.

BEFORE YOU START

A parabola, an ellipse and a hyperbola are the curves a plane cuts from a cone, and they are also the curves of degree two: every equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is one of them, or a degenerate case. Each one has a definition by distances, with a focus and either a directrix or a second focus, and a standard form that shows its center, its axes and its foci at a glance. These ten problems go from the standard forms to the general equation: completing the square to find the center, the discriminant to classify a conic with an xy term, and a rotation to remove it. They end with the calculus of conics, a tangent line, the polar form that describes orbits, and the area of an ellipse.

- **Parabola.** $y^2 = 4px$ has its focus at $(p, 0)$ and directrix $x = -p$; every point is as far from the focus as from the directrix. $x^2 = 4py$ opens up or down, with focus $(0, p)$ and directrix $y = -p$.
- **Ellipse.** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with $a > b > 0$ has vertices $(\pm a, 0)$, foci $(\pm c, 0)$ with $c^2 = a^2 - b^2$, and eccentricity $e = c/a < 1$. The distances to the two foci add up to $2a$.
- **Hyperbola.** $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ has vertices $(\pm a, 0)$, foci $(\pm c, 0)$ with $c^2 = a^2 + b^2$, asymptotes $y = \pm \frac{b}{a} x$ and eccentricity $e = c/a > 1$. The distances to the foci differ by $2a$. For $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$ the vertices are $(0, \pm a)$ and the asymptotes $y = \pm \frac{a}{b} x$.
- **Shifted center.** Replace x by $x - h$ and y by $y - k$ to move the center to (h, k). Complete the square to find h and k.
- **Classification.** For $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, the discriminant $B^2 - 4AC$ is negative for an ellipse, zero for a parabola and positive for a hyperbola, unless the conic is degenerate. Rotating the axes by θ with $\cot 2\theta = (A - C)/B$ removes the xy term.

- **Polar form.** With a focus at the origin and directrix $x = d$, a conic is $r = \dfrac{ed}{1 + e\cos\theta}$, where e is its eccentricity.

Builds on: Parametric and polar curves: slopes, arc length and area, Implicit differentiation: differentiate both sides, then solve for dy/dx

Problems

1. For the parabola $y^2 = 12x$, find the focus, the directrix and the length of the chord through the focus perpendicular to the axis (the latus rectum). Check that $(3, 6)$ is the same distance from the focus as from the directrix. difficulty ·

2. For the ellipse $9x^2 + 25y^2 = 225$, find a, b, c, the vertices, the foci and the eccentricity. Check that the point $(3, \frac{12}{5})$ has distances to the foci that add up to $2a$. difficulty ·

3. For the hyperbola $16x^2 - 9y^2 = 144$, find a, b, c, the vertices, the foci, the asymptotes and the eccentricity. difficulty ··

4. Put $x^2 + 4y^2 - 6x + 16y + 21 = 0$ in standard form. Find the center, the semi-axes and the foci. difficulty ··

5. Find the equation of the ellipse with foci $(\pm 3, 0)$ and major axis of length 10. Then find the hyperbola with vertices $(0, \pm 2)$ and asymptotes $y = \pm\frac{1}{2}x$. difficulty ··

6. Classify each conic: (a) $x^2 + xy + y^2 = 3$, (b) $x^2 + 4xy + y^2 = 3$, (c) $x^2 + 2xy + y^2 + x = 0$. difficulty ··

7. Rotate the axes to remove the xy term from $x^2 + xy + y^2 = 3$. Find the semi-axes and the directions of the major and minor axes. difficulty ···

8. Find the tangent line to $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$ at $(3, \frac{16}{5})$, by implicit differentiation. difficulty ··

9. Identify the conic $r = \dfrac{6}{2 + \cos\theta}$. Find its eccentricity, its directrix, its vertices and its equation in x and y.

difficulty ···

10. Find the area enclosed by the ellipse $9x^2 + 25y^2 = 225$ by integration, and check it against πab.

difficulty ···

Answers

1. Focus $(3, 0)$, directrix $x = -3$, latus rectum $12 = 4p$
2. $a = 5$, $b = 3$, $c = 4$; vertices $(\pm 5, 0)$, foci $(\pm 4, 0)$, $e = 0.8$; the distances add up to $10 = 2a$
3. Vertices $(\pm 3, 0)$, foci $(\pm 5, 0)$, asymptotes $y = \pm \frac{4}{3}x$, $e = \frac{5}{3}$
4. Center $(3, -2)$, $a = 2$, $b = 1$, $c = \sqrt{3}$; foci $(3 \pm \sqrt{3}, -2)$
5. Ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$; hyperbola $\dfrac{y^2}{4} - \dfrac{x^2}{16} = 1$
6. (a) ellipse, (b) hyperbola, (c) parabola
7. Semi-axes $\sqrt{6}$ along $y = -x$ (the v direction) and $\sqrt{2}$ along $y = x$
8. $y = 5 - \frac{3}{5}x$, the same as $\frac{3x}{25} + \frac{y}{5} = 1$
9. An ellipse with $e = \frac{1}{2}$, directrix $x = 6$: $\dfrac{(x+2)^2}{16} + \dfrac{y^2}{12} = 1$
10. Area $= 15\pi \approx 47.12 = \pi \cdot 5 \cdot 3$

Worked solutions

Problem 1

> For the parabola $y^2 = 12x$, find the focus, the directrix and the length of the chord through the focus perpendicular to the axis (the latus rectum). Check that $(3, 6)$ is the same distance from the focus as from the directrix.

$4p = 12$, so $p = 3$	Match $y^2 = 4px$.
Focus $(3, 0)$, directrix $x = -3$	$(p, 0)$ and $x = -p$.
At $x = 3$: $y^2 = 36$, so $y = \pm 6$	The chord through the focus perpendicular to the x-axis.
$(3, 6)$: distance to the focus 6, to the directrix $3 + 3 = 6$	The focus is directly below the point; the directrix is a vertical line.
Focus $(3, 0)$, **directrix** $x = -3$, **latus rectum** $12 = 4p$	The defining property holds at every point of the curve, which is what makes a parabolic mirror send every ray parallel to the axis through the focus.

Problem 2

> For the ellipse $9x^2 + 25y^2 = 225$, find a, b, c, the vertices, the foci and the eccentricity. Check that the point $(3, \frac{12}{5})$ has distances to the foci that add up to $2a$.

$\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$	Divide by 225.
$a = 5$, $b = 3$	The square roots of the denominators; the larger goes with x, so the major axis is horizontal.
$c = \sqrt{25 - 9} = 4$	For an ellipse $c^2 = a^2 - b^2$.
$\sqrt{7^2 + 2.4^2} + \sqrt{1^2 + 2.4^2} = 7.4 + 2.6$	Distances from $(3, 2.4)$ to $(-4, 0)$ and to $(4, 0)$.

$a = 5$, $b = 3$, $c = 4$; **vertices** $(\pm 5, 0)$, **foci** $(\pm 4, 0)$, $e = 0.8$; **the distances add up to** $10 = 2a$	The point is on the ellipse: $9 \cdot 9 + 25 \cdot 5.76 = 225$. Using $c^2 = a^2 + b^2$ instead puts the foci outside the ellipse (Mistake 1).

Problem 3

> For the hyperbola $16x^2 - 9y^2 = 144$, find a, b, c, the vertices, the foci, the asymptotes and the eccentricity.

$\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$	Divide by 144.
$a = 3$, $b = 4$	The square roots of 9 and 16; the positive term is in x, so it opens left and right.
$c = \sqrt{9 + 16} = 5$	For a hyperbola $c^2 = a^2 + b^2$.
Asymptotes $y = \pm \frac{4}{3} x$	$\pm b/a$: for large x the 1 on the right is negligible, so $16x^2 \approx 9y^2$.
Vertices $(\pm 3, 0)$, **foci** $(\pm 5, 0)$, **asymptotes** $y = \pm \frac{4}{3} x$, $e = \frac{5}{3}$	Check at $(5, \frac{16}{3})$, which is on the curve: its distances to the foci are $\frac{34}{3}$ and $\frac{16}{3}$, differing by $6 = 2a$.

Problem 4

> Put $x^2 + 4y^2 - 6x + 16y + 21 = 0$ in standard form. Find the center, the semi-axes and the foci.

$(x^2 - 6x) + 4(y^2 + 4y) = -21$	Group the x terms and the y terms; factor the 4 out of the y terms.
$(x - 3)^2 - 9 + 4\big((y + 2)^2 - 4\big) = -21$	Complete each square inside its own parentheses.
$(x - 3)^2 + 4(y + 2)^2 = -21 + 9 + 16 = 4$	The -4 inside the parentheses is multiplied by 4, so 16 moves to the right.

$\dfrac{(x-3)^2}{4} + \dfrac{(y+2)^2}{1} = 1$	Divide by 4.
Center $(3, -2)$, $a = 2$, $b = 1$, $c = \sqrt{3}$; **foci** $(3 \pm \sqrt{3}, -2)$	A horizontal ellipse. Moving only 4 instead of 16 to the right gives -8, an equation with no graph at all (Mistake 3).

Problem 5

> Find the equation of the ellipse with foci $(\pm 3, 0)$ and major axis of length 10. Then find the hyperbola with vertices $(0, \pm 2)$ and asymptotes $y = \pm \frac{1}{2}x$.

Ellipse: $2a = 10$, so $a = 5$; $c = 3$	The major axis runs between the vertices.
$b^2 = a^2 - c^2 = 25 - 9 = 16$	From $c^2 = a^2 - b^2$.
Hyperbola: vertices on the y-axis, so $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$ with $a = 2$	The positive term goes with the axis the vertices are on.
Asymptotes $y = \pm \frac{a}{b}x$, so $\frac{2}{b} = \frac{1}{2}$ and $b = 4$	For a vertical hyperbola the slope is a/b.
Ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$; **hyperbola** $\dfrac{y^2}{4} - \dfrac{x^2}{16} = 1$	Check the asymptotes: for large $\lvert x \rvert$, $y^2 \approx 4 \cdot x^2/16$, so $y \approx \pm x/2$. Taking the slope as b/a gives $b = 1$ and asymptotes $y = \pm 2x$ (Mistake 2).

Problem 6

> Classify each conic: (a) $x^2 + xy + y^2 = 3$, (b) $x^2 + 4xy + y^2 = 3$, (c) $x^2 + 2xy + y^2 + x = 0$.

(a) $B^2 - 4AC = 1 - 4 = -3 < 0$	$A = 1$, $B = 1$, $C = 1$.
(b) $B^2 - 4AC = 16 - 4 = 12 > 0$	$A = 1$, $B = 4$, $C = 1$.
(c) $B^2 - 4AC = 4 - 4 = 0$	$A = 1$, $B = 2$, $C = 1$.

(c) $(x+y)^2 = -x$	The quadratic part is a perfect square, the mark of a parabola.
(a) ellipse, (b) hyperbola, (c) parabola	In (b) both squared terms are positive, but the xy term outweighs them: along $y=-x$ the left side is $-2x^2$, which takes the curve's equation negative (Mistake 5).

Problem 7

> Rotate the axes to remove the xy term from $x^2 + xy + y^2 = 3$. Find the semi-axes and the directions of the major and minor axes.

$\cot 2\theta = (A-C)/B = 0$, so $\theta = 45^\circ$	$A = C$.
$x = \frac{u-v}{\sqrt{2}}$, $y = \frac{u+v}{\sqrt{2}}$	The rotation by 45°.
$x^2 + y^2 = u^2 + v^2$ and $xy = \frac{1}{2}(u^2 - v^2)$	Expand both; the uv terms cancel.
$\frac{3}{2}u^2 + \frac{1}{2}v^2 = 3$, that is $\dfrac{u^2}{2} + \dfrac{v^2}{6} = 1$	Substitute and divide by 3.
Semi-axes $\sqrt{6}$ along $y=-x$ (the v direction) and $\sqrt{2}$ along $y=x$	Check: on $y=-x$, $x^2=3$, so $x=\sqrt{3}$ and the distance is $\sqrt{6}$; on $y=x$, $3x^2=3$ and the distance is $\sqrt{2}$. The coefficients $\frac{3}{2}$ and $\frac{1}{2}$ are the eigenvalues of $\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$.

Problem 8

> Find the tangent line to $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$ at $(3, \frac{16}{5})$, by implicit differentiation.

$\frac{2x}{25} + \frac{2y}{16}\,y' = 0$	Differentiate both sides in x; y is a function of x.
$y' = -\dfrac{16x}{25y}$	Solve for y'.

At $(3, \frac{16}{5})$: $y' = -\dfrac{48}{80} = -\frac{3}{5}$	$25 \cdot \frac{16}{5} = 80$.
$y - \frac{16}{5} = -\frac{3}{5}(x - 3)$	Point-slope form.
$y = 5 - \frac{3}{5}x$, **the same as** $\frac{3x}{25} + \frac{y}{5} = 1$	The general tangent to an ellipse at (x_0, y_0) is $\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$.

Problem 9

> Identify the conic $r = \dfrac{6}{2 + \cos\theta}$. Find its eccentricity, its directrix, its vertices and its equation in x and y.

$r = \dfrac{3}{1 + \frac{1}{2}\cos\theta}$	Divide the top and bottom by 2 to get the form $ed/(1 + e\cos\theta)$.
$e = \frac{1}{2}$, so an ellipse; $ed = 3$ gives $d = 6$	Directrix $x = 6$.
$r(0) = 2$ and $r(\pi) = 6$	The vertices are $(2, 0)$ and $(-6, 0)$.
$2a = 8$, center $(-2, 0)$, $c = 2$, $b^2 = 16 - 4 = 12$	The center is halfway between the vertices, and the focus at the origin is $c = 2$ from it.
An ellipse with $e = \frac{1}{2}$, directrix $x = 6$: $\dfrac{(x+2)^2}{16} + \dfrac{y^2}{12} = 1$	Check: $c/a = 2/4 = e$. This is the shape of a planet's orbit, with the sun at the focus at the origin.

Problem 10

> Find the area enclosed by the ellipse $9x^2 + 25y^2 = 225$ by integration, and check it against πab.

Upper half: $y = 3\sqrt{1 - x^2/25}$	Solve $\frac{x^2}{25} + \frac{y^2}{9} = 1$ for $y \geq 0$.
Area $= 4\int_0^5 3\sqrt{1 - x^2/25}\,dx$	Four equal quarters, by symmetry.
$x = 5\sin t$, $dx = 5\cos t\,dt$	Trig substitution; t runs from 0 to $\frac{\pi}{2}$.

$= 60 \int_0^{\pi/2} \cos^2 t \, dt = 60 \cdot \frac{\pi}{4}$	$\sqrt{1 - \sin^2 t} = \cos t$ on this interval.
Area $= 15\pi \approx 47.12 = \pi \cdot 5 \cdot 3$	In general the ellipse is the unit circle stretched by a in x and b in y, which multiplies its area π by ab.

Where this goes wrong

1. Using c² = a² + b² for an ellipse

Ellipses and hyperbolas both have a, b and c, and the hyperbola's rule is the one that looks like Pythagoras.

$\frac{x^2}{25} + \frac{y^2}{9} = 1$, $a = 5$, $b = 3$	Right so far: Problem 2.
"$c^2 = a^2 + b^2$, like a right triangle."	For an ellipse the right triangle has a as its hypotenuse: from a focus to the end of the minor axis is a, with legs b and c.
$c = \sqrt{34} \approx 5.83$, foci $(\pm 5.83, 0)$ ✗	That is past the vertices at ± 5, outside the ellipse. The foci are inside: $c = \sqrt{25 - 9} = 4$.

2. Taking the asymptote slope upside down

The rule $y = \pm \frac{b}{a} x$ is remembered without the hyperbola it was stated for.

$\frac{y^2}{4} - \frac{x^2}{b^2} = 1$, vertices $(0, \pm 2)$, asymptotes $y = \pm \frac{1}{2} x$	Right so far: Problem 5.
"The slope of the asymptotes is b/a."	That is for $x^2/a^2 - y^2/b^2 = 1$. For a hyperbola opening up and down, setting the right side to 0 gives $y = \pm \frac{a}{b} x$.
$\frac{b}{2} = \frac{1}{2}$, so $b = 1$ and the hyperbola is $\frac{y^2}{4} - x^2 = 1$ ✗	Its asymptotes are $y = \pm 2x$, four times too steep. The right answer is $b = 4$. Check any answer by setting the 1 to 0 and solving for y.

3. Completing the square without the coefficient

The coefficient 4 is factored out of the y terms in Problem 4, and the number that completes the square is added inside the parentheses.

$(x-3)^2 - 9 + 4(y^2 + 4y + 4 - 4) = -21$	Right so far: adding and subtracting 4 inside changes nothing.
"Move the 9 and the 4 to the right side."	The -4 is inside parentheses that are multiplied by 4, so it is really -16.
$(x-3)^2 + 4(y+2)^2 = -21 + 9 + 4 = -8$ ✗	A sum of squares cannot be negative, so this says the equation has no graph. The right side is $-21 + 9 + 16 = 4$, an ellipse.

4. Reading a and b without the square roots

In Problem 3, the denominators 9 and 16 are the numbers on the page.

$\frac{x^2}{9} - \frac{y^2}{16} = 1$	Right so far.
"$a = 9$ and $b = 16$."	The denominators are a^2 and b^2. Setting $y = 0$ gives $x^2 = 9$, not $x = 9$.
Vertices $(\pm 9, 0)$, asymptotes $y = \pm\frac{16}{9}x$ ✗	The vertices are $(\pm 3, 0)$ and the asymptotes $y = \pm\frac{4}{3}x$. Plug a vertex back in: $81/9 = 9$, not 1.

5. Classifying by the signs of x² and y² alone

Without an xy term, both squares positive means an ellipse, and the rule is applied to Problem 6(b).

$x^2 + 4xy + y^2 = 3$, with $A = 1$ and $C = 1$	Right so far.
"Both squared terms are positive, so it is an ellipse."	The sign rule is the discriminant test with $B = 0$. With an xy term, $B^2 - 4AC = 16 - 4 = 12 > 0$.

An ellipse ✗	A hyperbola: rotating by 45° gives $3u^2 - v^2 = 3$. An ellipse would be bounded, but the points $(t, t(\sqrt{3}-2))$ satisfy $x^2 + 4xy + y^2 = 0$ for every t, and the curve runs alongside them to infinity.

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